

# Chapter 6. Extending the ARMA model: Seasonality and trend

## Objectives

- ① Monthly time series often exhibit seasonal variation. January data are similar to observations at a different January, etc.
- ② Many time series exhibit a trend.
- ③ We wish to extend the theoretical and practical elegance of the ARMA framework to cover these situations.

# Seasonal autoregressive moving average (SARMA) models

- A general  $\text{SARMA}(p, q) \times (P, Q)_{12}$  model for monthly data is

$$[S1] \quad \phi(B)\Phi(B^{12})(Y_n - \mu) = \psi(B)\Psi(B^{12})\epsilon_n,$$

where  $\{\epsilon_n\}$  is a white noise process and

$$\begin{aligned}\mu &= \mathbb{E}[Y_n] \\ \phi(x) &= 1 - \phi_1 x - \cdots - \phi_p x^p, \\ \psi(x) &= 1 + \psi_1 x + \cdots + \psi_q x^q, \\ \Phi(x) &= 1 - \Phi_1 x - \cdots - \Phi_P x^P, \\ \Psi(x) &= 1 + \Psi_1 x + \cdots + \Psi_Q x^Q.\end{aligned}$$

- A SARMA model is a special case of an ARMA model, where the AR and MA polynomials are factored into a **monthly** polynomial in  $B$  and an **annual** polynomial in  $B^{12}$ . The annual polynomial is also called the **seasonal** polynomial.
- Thus, everything we learned about ARMA models (including assessing causality, invertibility and reducibility) also applies to SARMA.

# Choosing the period for a SARMA model

- For the  $\text{SARMA}(p, q) \times (P, Q)_{12}$  model, 12 is called the **period**.
- One could write a SARMA model for some period other than 12.
- A  $\text{SARMA}(p, q) \times (P, Q)_4$  model could be appropriate for quarterly data.
- In principle, a  $\text{SARMA}(p, q) \times (P, Q)_{52}$  model could be appropriate for weekly data, though in practice ARMA and SARMA may not work so well for higher frequency data.
- The seasonal period should be appropriate for the system being modeled. It is usually inappropriate to fit a  $\text{SARMA}(p, q) \times (P, Q)_9$  model just because you notice a high sample autocorrelation at lag 9.

Consider the following two models:

$$[S2] \quad Y_n = 0.5Y_{n-1} + 0.25Y_{n-12} + \epsilon_n,$$

$$[S3] \quad Y_n = 0.5Y_{n-1} + 0.25Y_{n-12} - 0.125Y_{n-13} + \epsilon_n,$$

**Question 6.1.** Which of [S2] and/or [S3] is a SARMA model?

[S3] is  $(1 - 0.5B)(1 - 0.25B^{12})Y_n = \epsilon_n$   
so it is SARMA.

[S2] lacks the  $B \cdot B^{12}$  term so is not SARMA.

**Question 6.2.** Why do we assume a multiplicative structure in [S1]?

What theoretical and practical advantages (or disadvantages) arise from requiring that an ARMA model for seasonal behavior has polynomials that can be factored as a product of a monthly polynomial and an annual polynomial? The multiplicative form adds simplicity & makes it easier to find roots. This is not necessarily related to being a good model for data.

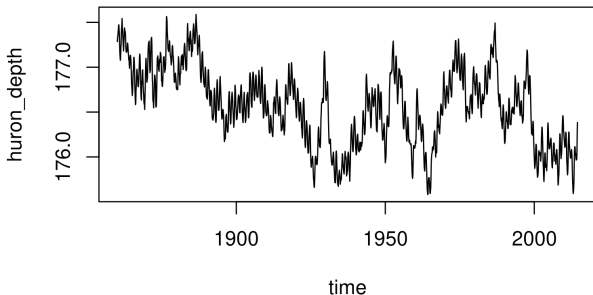
# Fitting a SARMA model

Let's do this for the full, monthly, version of the Lake Huron depth data described earlier.

```
head(dat,3)
```

```
##           Date Average year month
## 1 1860-01-01 177.285 1860      1
## 2 1860-02-01 177.339 1860      2
## 3 1860-03-01 177.349 1860      3
```

```
huron_depth <- dat$Average
time <- dat$year + dat$month/12
# Note: we treat December 2011 as time 2012.0, etc
plot(huron_depth~time,type="l")
```



Now, we get to fit a model. Based on our previous analysis, we'll go with AR(1) for the annual polynomial. Let's try ARMA(1,1) for the monthly part. In other words, we seek to fit the model

$$(1 - \Phi_1 B^{12})(1 - \phi_1 B)Y_n = (1 + \psi_1 B)\epsilon_n.$$

```
huron_sarma11x10 <- arima(huron_depth,  
  order=c(1,0,1),  
  seasonal=list(order=c(1,0,0),period=12)  
)
```

```
huron_sarma11x10
```

```
##
```

```
## Call:
```

```
## arima(x = huron_depth, order = c(1, 0, 1), seasonal = list(order  
##      period = 12))
```

```
##
```

```
## Coefficients:
```

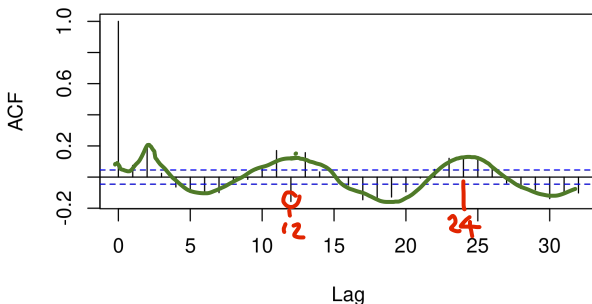
	ar1	ma1	sar1	intercept
##	0.9641	0.3782	0.5104	176.5714
## s.e.	0.0063	0.0203	0.0218	0.0909

```
##
```

```
## sigma^2 estimated as 0.002592:  log likelihood = 2884.36,  aic =
```

# Residual analysis

- Residual analysis is similar to non-seasonal ARMA models.
- We look for residual correlations at lags corresponding to multiples of the period (here, 12, 24, 36, ...) for misspecified annual dependence.



**Question 6.3.** What do you conclude from this residual analysis? What would you do next?

Perhaps try an extra seasonal term, say  $(p, q) = (1, 1)$  not  $(1, 0)$ . The oscillatory component for within-year variation might suggest  $p=2$ , since  $p=1$  can only give exponentially decaying ACF.



# ARMA models for differenced data

- Applying a difference operation to the data can make it look more stationary and therefore more appropriate for ARMA modeling.
- This can be viewed as a **transformation to stationarity**
- We can transform the data  $y_{1:N}$  to  $z_{2:N}$

$$z_n = \Delta y_n = y_n - y_{n-1}.$$

- Then, an ARMA(p,q) model  $Z_{2:N}$  for the differenced data  $z_{2:N}$  is called an **integrated autoregressive moving average** model for  $y_{1:N}$  and is written as ARIMA(p,1,q).
- Formally, the ARIMA(p,d,q) model with intercept  $\mu$  for  $Y_{1:N}$  is

$$[S4] \quad \phi(B)((1-B)^d Y_n - \mu) = \psi(B)\epsilon_n,$$

where  $\{\epsilon_n\}$  is a white noise process;  $\phi(x)$  and  $\psi(x)$  are ARMA polynomials.

- It is unusual to fit an ARIMA model with  $d > 1$ .

- We see that an  $ARIMA(p,1,q)$  model is almost a special case of an  $ARMA(p+1,q)$  model with a **unit root** to the  $AR(p+1)$  polynomial.

**Question 6.4.** Why “almost” not “exactly” in the previous statement?

*Note the different treatments of the intercept,  $\mu$ .*

## Two reasons to fit an ARIMA( $p,d,q$ ) model with $d > 0$

1. You may really think that modeling the differences is a natural approach for your data. The S&P 500 stock market index analysis in Chapter 3 is an example of this, as long as you remember to first apply a logarithmic transform to the data.
2. Differencing often makes data look “more stationary” and perhaps it will then look stationary enough to justify applying the ARMA machinery.
  - We should be cautious about this second reason. It can lead to poor model specifications and hence poor forecasts or other conclusions.
  - The second reason was more compelling in the 1970s and 1980s. Limited computing power resulted in limited alternatives, so it was practical to force as many data analyses as possible into the ARMA framework and use method of moments estimators.

# Practical advice on using ARIMA models

- ARIMA analysis is relatively simple to do. It has been a foundational component of time series analysis since the publication of the influential book “Time Series Analysis” by Box and Jenkins (1st edition, 1970) which developed and popularized ARIMA modeling.
- A practical approach is:
  1. Do a competent ARIMA analysis.
  2. Identify potential limitations in this analysis and remedy them using more advanced methods.
  3. Assess whether you have in fact learned anything from (2) that goes beyond (1).

# The SARIMA( $p, d, q$ ) $\times$ ( $P, D, Q$ ) model

Combining integrated ARMA models with seasonality, we can write a general SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ )<sub>12</sub> model for nonstationary monthly data, given by

$$\begin{aligned} \text{[S5]} \quad & \phi(B)\Phi(B^{12})((1-B)^d(1-B^{12})^DY_n - \mu) \\ & = \psi(B)\Psi(B^{12})\epsilon_n, \end{aligned}$$

where  $\{\epsilon_n\}$  is a white noise process, the intercept  $\mu$  is the mean of the differenced process  $\{(1-B)^d(1-B^{12})^DY_n\}$ , and we have ARMA polynomials  $\phi(x)$ ,  $\Phi(x)$ ,  $\psi(x)$ ,  $\Psi(x)$  as in model [S1].

- The SARIMA(0, 1, 1)  $\times$  (0, 1, 1)<sub>12</sub> model has often been used for forecasting monthly time series in economics and business. It is sometimes called the **airline model** after a data analysis by Box and Jenkins (1970).

# Modeling trend with ARMA noise

- A general **signal plus noise** model is

$$[S6] \quad Y_n = \mu_n + \eta_n,$$

where  $\{\eta_n\}$  is a stationary, mean zero stochastic process, and  $\mu_n$  is the mean function.

- If, in addition,  $\{\eta_n\}$  is uncorrelated, then we have a **signal plus white noise** model. The usual linear trend regression model fitted by least squares in Chapter 2 corresponds to a signal plus white noise model.
- We can say **signal plus colored noise** if we wish to emphasize that we're not assuming white noise.
- Here, **signal** and **trend** are used interchangeably. In other words, we are assuming a deterministic signal.
- At this point, it is natural for us to consider a signal plus ARMA(p,q) noise model, where  $\{\eta_n\}$  is a stationary, causal, invertible ARMA(p,q) process with mean zero.
- As well as the  $p + q + 1$  parameters in the ARMA(p,q) model, there will usually be unknown parameters in the mean function.

# Linear regression with ARMA errors

- When the mean function (also known as the trend) has a linear specification,

$$\mu_n = \sum_{k=1}^K Z_{n,k} \beta_k,$$

the signal plus ARMA noise model is known as **linear regression with ARMA errors**.

- Writing  $Y$  for a column vector of  $Y_{1:N}$ ,  $\mu$  for a column vector of  $\mu_{1:N}$ ,  $\eta$  for a column vector of  $\eta_{1:N}$ , and  $Z$  for the  $N \times K$  matrix with  $(n, k)$  entry  $Z_{n,k}$ , we have a general linear regression model with correlated ARMA errors,

$$Y = Z\beta + \eta.$$

$\eta = Y - Z\beta$   
so for a given  $\beta$ ,  
we can calculate the likelihood of  
 $\eta$  using `arima()`.

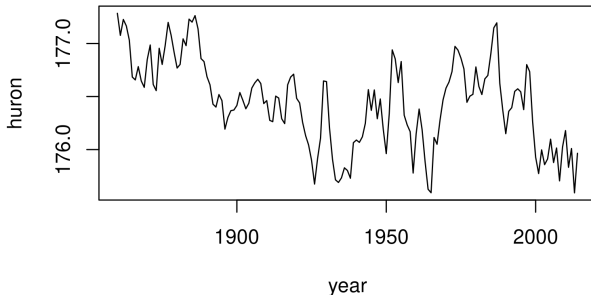
# Inference for the linear regression model with ARMA errors

- Maximum likelihood estimation of  $\theta = (\phi_{1:p}, \psi_{1:q}, \sigma^2, \beta)$  is a nonlinear optimization problem.
- Fortunately, `arima` in R can do it for us.
- As usual, we should look out for signs of numerical problems.
- Data analysis for a linear regression with ARMA errors model, using the framework of likelihood-based inference, is procedurally similar to fitting an ARMA model.
- This is a powerful technique, since the covariate matrix  $Z$  can include other time series. We can evaluate associations between different time series.
- With appropriate care (since **association is not causation**) we can draw inferences about mechanistic relationships between dynamic processes.



# Evidence for systematic trend in Lake Huron depth?

Let's go back to annual data, say the January depth, to avoid seasonality.



- Visually, there seems some evidence for a decreasing trend, but there are also considerable fluctuations.
- Let's test for a trend, using a regression model with Gaussian AR(1) errors. We have previously found that this is a reasonable model for these data.
- First, let's fit a null model.

```
fit0 <- arima(huron,order=c(1,0,0))
```

```
## Call:
## arima(x = huron, order = c(1, 0, 0))
##
## Coefficients:
##          ar1  intercept
##       0.8694   176.4588
## s.e.  0.0407     0.1234
##
## sigma^2 estimated as 0.04368:  log likelihood = 22,  aic = -38
```

note: R has  
dropped 2 trailing  
zeros. This is  
really 22.00 (2 d.p.)

```
fit1 <- arima(huron, order=c(1,0,0), xreg=year)
```

to avoid this in  
your code, use  
`myround()` from the `brman`  
R package.

```
##
## Call:
## arima(x = huron, order = c(1, 0, 0), xreg = year)
##
## Coefficients:
##          ar1  intercept      year
##       0.8240   186.0146  -0.0049
## s.e.  0.0451     3.7417   0.0019
##
## sigma^2 estimated as 0.0423:  log likelihood = 24.62,  aic = -41
```

✳ this is how to  
put in covariates for  
regression with  
ARMA errors.

# Setting up a formal hypothesis test

- To talk formally about these results, we must down a model and some hypotheses.
- Writing the data as  $y_{1:N}$ , collected at years  $t_{1:N}$ , the model we have fitted is

$$(1 - \phi_1 B)(Y_n - \mu - \beta t_n) = \epsilon_n,$$

where  $\{\epsilon_n\}$  is Gaussian white noise with variance  $\sigma^2$ . Our null model is

$$H^{\langle 0 \rangle} : \beta = 0,$$

and our alternative hypothesis is

$$H^{\langle 1 \rangle} : \beta \neq 0.$$

**Question 6.5.** How do we test  $H^{(0)}$  against  $H^{(1)}$ ?

- Construct two different tests using the R output above.
- Which test do you prefer, and why?
- How would you check whether your preferred test is indeed better?

1. Z test on the coefficient:

test statistic,  $\hat{\beta} = -0.0049$

Z statistic,  $\hat{\beta} / SE(\hat{\beta}) = \hat{z}$  giving rise to a p-value

$\hat{z} > 1.96$ , so we reject  
 $H_0$  at 5% size.



2. Likelihood ratio test.

Difference in log likelihood =  $24.62 - 22.00 = 2.62$

This is larger than the 1.92 cutoff for a test at  
5% size

**Question 6.6.** What other supplementary analysis could you do to strengthen your conclusions?

1. plot residuals.

2. simulation study, e.g. bootstrap.

# Acknowledgments and License

- These notes build on previous versions at `ionides.github.io/531w16` and `ionides.github.io/531w18`.
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