Chapter 4. Probability models

- A **probability model** is an assignment of probabilities to possible outcomes.
- We don't observe these probabilities. We observe a particular dataset.
- If we treat the dataset as an outcome of a probability model, we can answer questions such as,

  "*If there really is no association between unemployment and life expectancy, what is the probability we would see a least squares linear model coefficient as large as the one we actually observed, due to random fluctuations in the data?*"

- Here, we are particularly interested in developing a probability model for the linear model.
- First, we need some basic tools for probability models: random variables, the normal distribution, mean, variance and standard deviation.

Random variables and events

- A **random variable** X is a random number with probabilities assigned to outcomes.

  Example: Let X be a roll of a fair die. A natural probability model is to assign probability of $1/6$ to each of the possible outcomes $1, 2, 3, 4, 5, 6$.

- An **event** is a set of possible outcomes.

  Example: For a die, $E = \{X \geq 4\} = \{4, 5, 6\}$ is the event that the die shows 4 or more.

- We can assign probabilities to events just like to outcomes.

  Example: For a die, $\mathrm{P}(E) = \mathrm{P}(X \geq 4) = 3/6 = 1/2$.

Question 4.1. If an experiment can be repeated many times (like rolling a die) how can you check whether the probability model is correct?

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Notation for combining events

- $\{E \text{ or } F\}$ is the event that either E or F or both happens.
- Since E and F are sets, we can write this as a union, $\{E \text{ or } F\} = E \cup F$
- $\{E \text{ and } F\}$ is the event that both E and F happen.
- We can write this as an intersection,

$$\{E \text{ and } F\} = E \cap F$$

- Usually, we prefer "and/or" to "intersection/union".

Question 4.2. When does this formal use of "and" and "or" agree with usual English usage? When does it disagree?

The basic rules of probability

1. Probabilities are numbers between 0 (impossible) and 1 (certain).
2. Let $\mathcal{S}$ be the set of all possible outcomes. Then, $\mathrm{P}(\mathcal{S}) = 1$.
   Example: For a die, $\mathrm{P}(X \in \{1, 2, 3, 4, 5, 6\}) = 1$.
3. Events E and F are called **mutually exclusive** if they cannot happen at the same time. In other words, their intersection is the empty set. In this case,

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F).$$

Question 4.3. You roll a red die and a blue die. Let $E = \{\text{red die shows 1}\}$, $F = \{\text{blue die shows 1}\}$, $G = \{\text{red die shows 6}\}$. (a) Are E and F mutually exclusive? (b) How about E and G? (c) How about F and G?

Discrete random variables

- X is a **discrete random variable** if we can list all its possible outcomes. Let's call them $x_1, x_2, \ldots$.
- A discrete random variable is specified by probability that the random variable takes each possible outcome,

$$p_i = \mathrm{P}[X = x_i], \text{for } i = 1, 2, 3, \ldots$$

- It can be helpful to plot a graph of p_i against x_i.
- This graph is called the **probability mass function**.

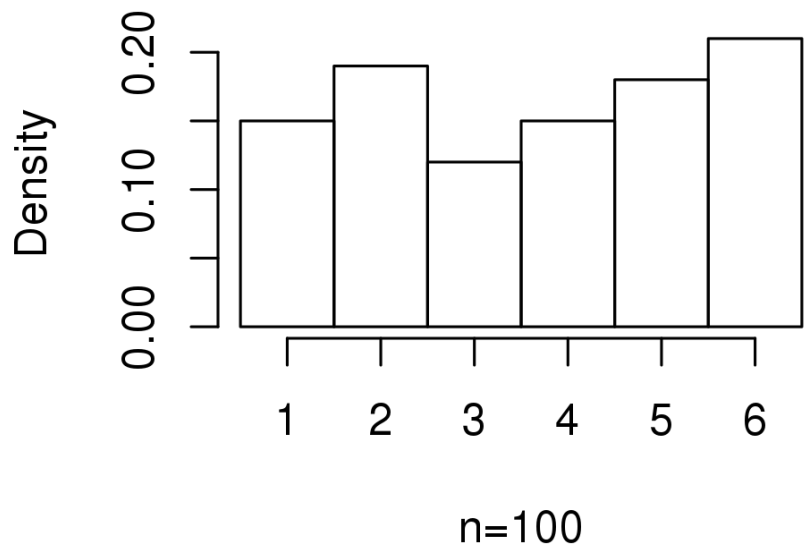
Question 4.4. Sketch the probability mass function for a fair die.

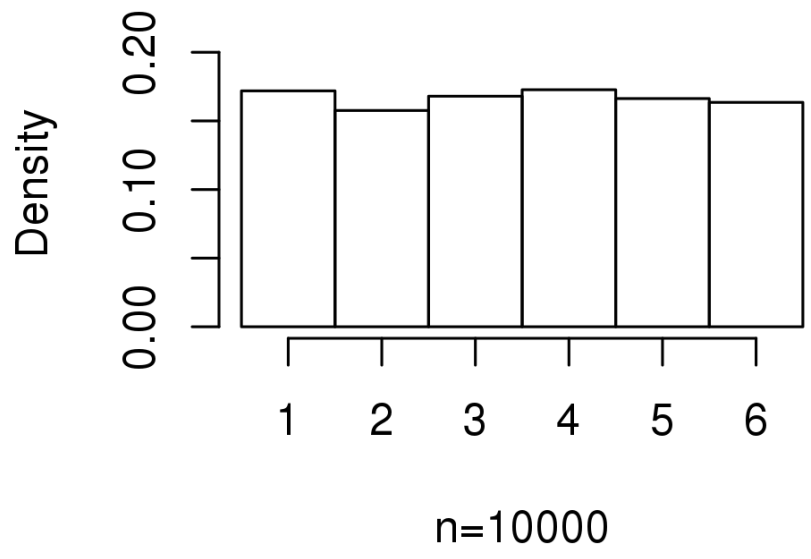
Simulating the law of large numbers

- The "law of large numbers" says that the proportion of each outcome i in a large number of draws of a discrete random variable approaches p_i.
- We can test this by simulation, using the `replicate()` command.

Worked example 4.1. In R, a random draw with replacement from $\{1, 2, 3, 4, 5, 6\}$ can be obtained by `sample(1:6,size=1)` This is equivalent to one roll of a fair die.

```
hist(replicate(n=100,sample(1:6,size=1) ),
  main="",prob=TRUE,breaks=0.5:6.5,xlab="n=100",ylim=c(0,0.21))
```





Continuous random variables: the normal distribution

- A **continuous random variable** is one which can take any value in an interval of the real numbers.

  Example: physical quantities such as time and speed are not limited to a discrete set of possible values.

- We will often see the **normal distribution**.
- Let's look at **normal random variables** simulated by R using `rnorm()`.

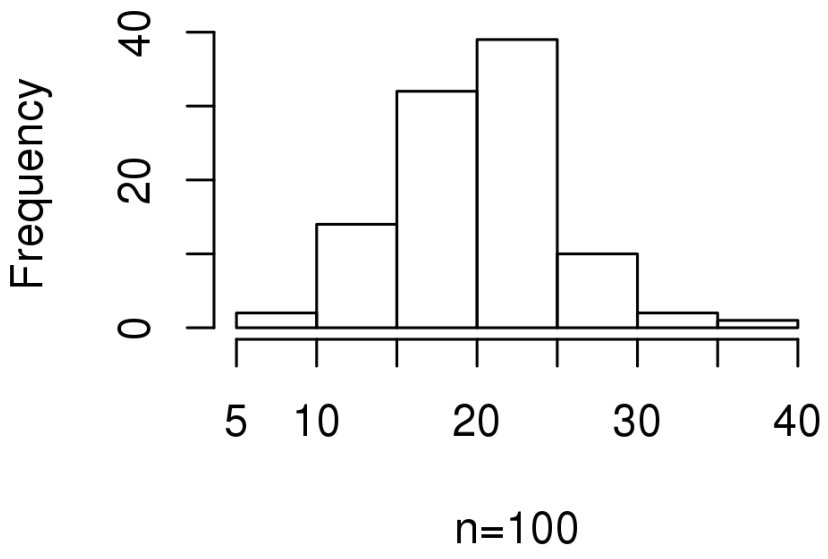
```
rnorm(n=10,mean=20,sd=5)
```

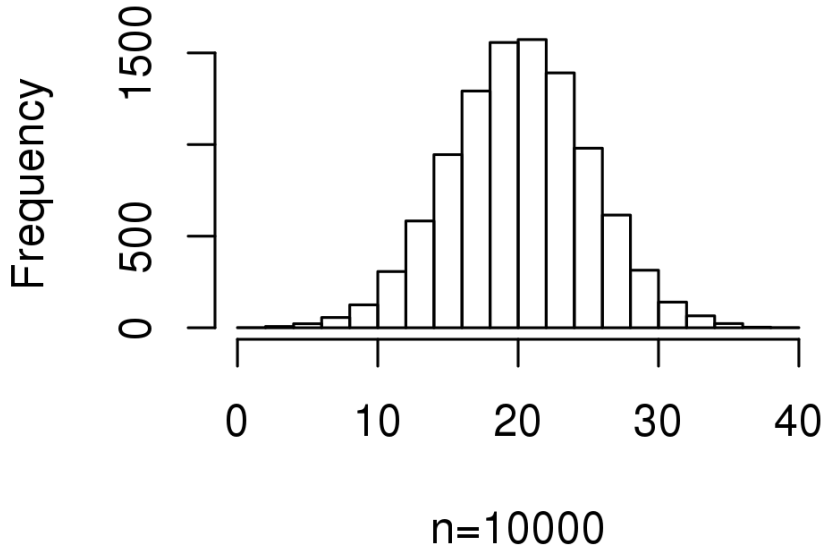
```
##  [1] 14.01141 26.18597 17.18972 21.12226 15.20211 30.34660
##  [7] 19.94277 22.05179 27.73804 25.94906
```

- The arguments `mean=20,sd=5` of `rnorm()` are the **parameters** of the normal distribution.
- A normal random variable can take any numeric value: it is continuous.
- Values are centered on the mean and are usually less than twice the standard deviation (sd) from the mean.

A histogram of normal distribution simulations

```
hist(rnorm(n=100,mean=20,sd=5),
  main="",xlab="n=100")
```





- Large samples from the normal distribution follow a **bell curve** histogram.
- From smaller samples, this is harder to see.

Finding probabilities for a continuous random variable

- A continuous random variable X has a **probability density function** $f(x)$ which is integrated to find the probability that X falls in any interval:

$$\mathrm{P}(a < X < b) = \int_a^b f(x)\,dx$$

- Write $X \sim \mathrm{normal}(\mu, \sigma)$ to mean X is a normal random variable with mean μ and sd σ. The probability density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

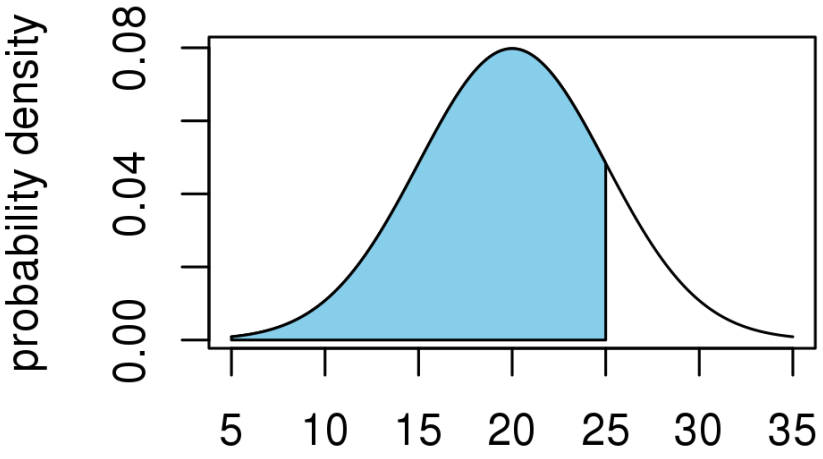
and so

$$\mathrm{P}(a < X < b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

- This integral has no closed form solution.
- Fortunately, R provides `pnorm()` and `qnorm()` that let us work with probabilities for the normal distribution numerically.

Calculating probabilities for the normal distribution

- `pnorm()` finds the **left tail** of the normal distribution.



Example: `pnorm(25,mean=20,sd=5)` computes the shaded area above.

- We don't have to do calculus with the normal integral, but we do use the relationship between the curve, area under the curve, and probability. And we must know how to compute these things in R.
- For $X \sim \text{normal}(\mu, \sigma)$,

$$\texttt{pnorm(x,mu,sigma)} = \text{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2}\, dy$$

Finding probabilities that are not a left tail

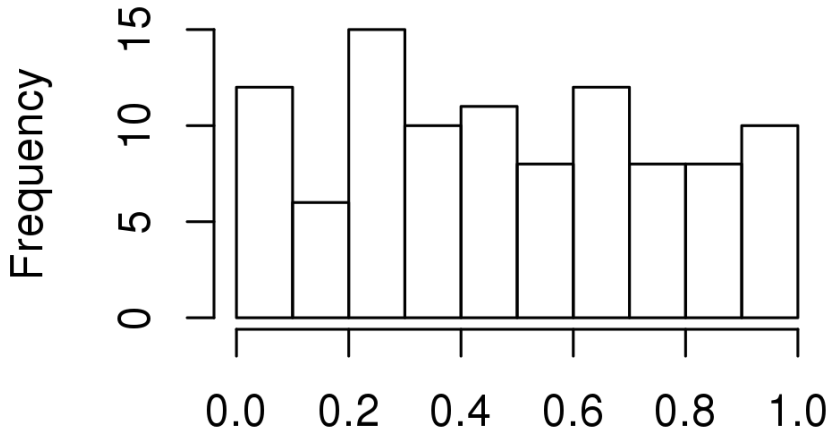
Question 4.5. Let $X \sim \text{normal}(10, 10)$. Sketch a shaded area under a curve giving $\text{P}(0 \leq X \leq 10)$. Write this probability as an integral and as R code.

Hint: To use `pnorm()`, think of the shaded area as the difference of two left tails.

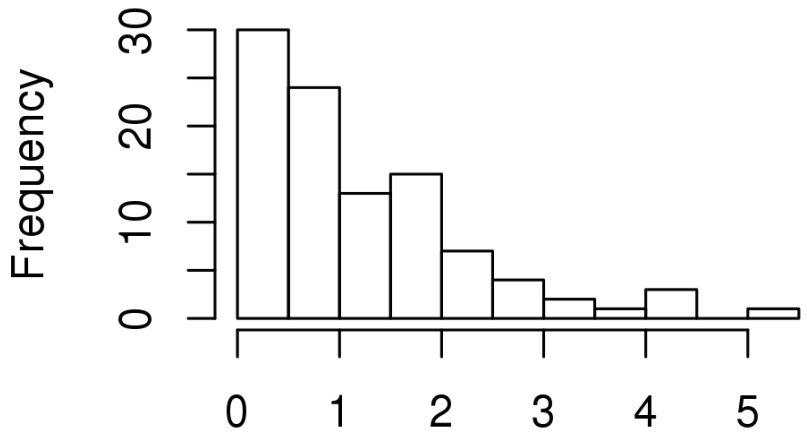
Other continuous distributions

- There are many continuous random variables that are not normally distributed.
- For example, we could explore the **uniform** or **exponential** distributions.

```
hist(runif(100))
```




```
hist(rexp(100))
```



- Normal random variables are the building block for all the random variables we work with in this class.
- Let's investigate why the normal distribution is so important.

Sums and averages follow the normal distribution

- When we sum many random quantities, the sum often follows a normal distribution even if each term in the sum is not normally distributed.
- This property is called **the central limit theorem**.
- It is an empirical fact as well as a mathematical theorem!
- Averaging is multiplying the sum by a constant ($1/n$). A bell curve is still a bell curve when we rescale by multiplication.

Question 4.6. Would you expect a histogram of student heights to follow a normal curve? Why? Why not?

Question 4.7. Would you expect a histogram of the daily change in the Dow Jones stock market index to follow a normal curve? Why? Why not?

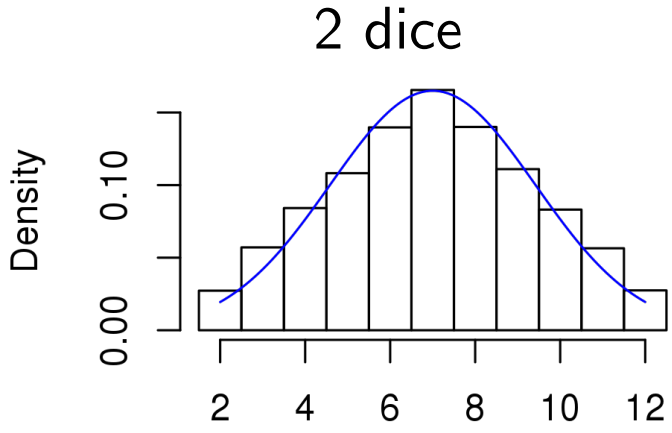
Demonstrating the central limit theorem with dice

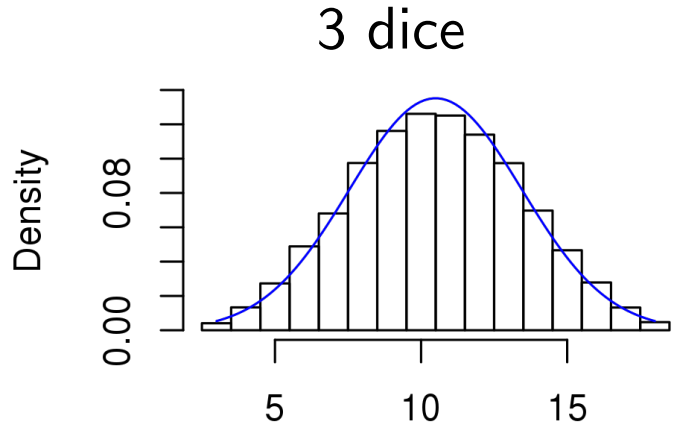
- `sample(1:6,2,replace=TRUE)` simulates the outcome of rolling two dice.
- `sum(sample(1:6,2,replace=TRUE))` simulates the sum of two dice.
- `replicate()` lets us see what happens if we do this many times

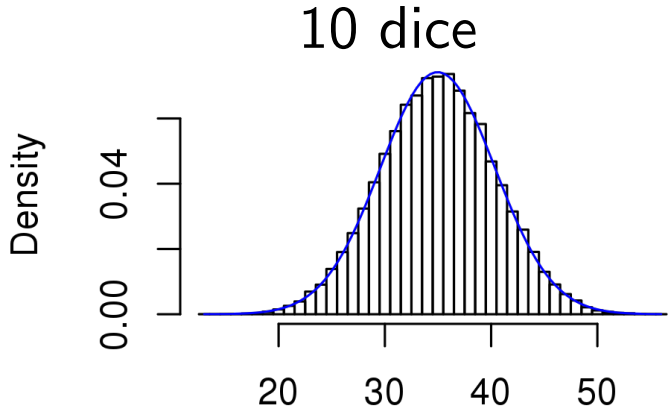
```
dice2 <- replicate(50000,sum(sample(1:6,2,replace=TRUE)))
dice3 <- replicate(50000,sum(sample(1:6,3,replace=TRUE)))
dice10 <- replicate(50000,sum(sample(1:6,10,replace=TRUE)))
dice20 <- replicate(50000,sum(sample(1:6,20,replace=TRUE)))
```

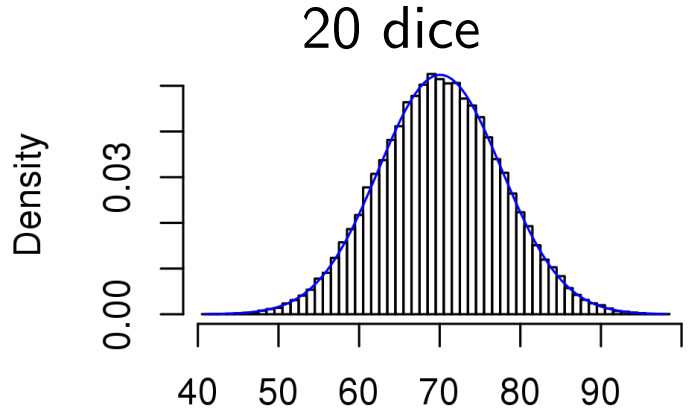
- Guess how many dice you have to add up before the histogram looks normal?

```
hist(dice2,prob=TRUE,breaks=(min(dice2)-0.5):(max(dice2)+0.5))
normal.x <- seq(from=min(dice2),to=max(dice2),length=100)
normal.y <- dnorm(normal.x,mean=mean(dice2),sd=sd(dice2))
lines(normal.x,normal.y,col="blue")
```









Question 4.8. Why do we use `prob` and `breaks` arguments to `hist()`?

More normal approximation situations

Question 4.9. Would you expect detrended data on life expectancy at birth to follow a normal distribution? Explain.

Question 4.10. Consider the mice weight data for HW1 with mice randomized to two treatments: a high fat diet and a usual lab diet. (a) Would you expect the weights of mice in each treatment group to follow a normal distribution? (b) If the experiment were replicated ten times, and an average weight calculated for each of these ten replications, would you expect the ten averages to follow a normal distribution? Are your answers different for (a) and (b)?

The sample mean and the expectation of a random variable

- The **sample mean** or **average** of $\mathbf{y} = (y_1, \ldots, y_n)$ is

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

- The **expected value** of a random variable X taking possible values $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$ is

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} x_i p_i$$

- If we have many draws of X, the sample proportion taking value x_i becomes close to p_i and so the sample mean becomes close to the expected value.
- If X is a continuous random variable with density $f(x)$ the sum for expected value becomes an integral,

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

The sample mean and expectation using R

- We've already seen the `mean()` function which computes the sample mean of a numeric vector.
- One way to compute the expected value of a random variable is to take the sample mean of many realizations of the random variable.

```
x <- rnorm(10000,mean=20,sd=5)
mean(x)
```

```
## [1] 19.96731
```

- The same calculation can be done using `replicate()`:

```
y <- replicate(n=10000,rnorm(1,mean=20,sd=5))
mean(y)
```

```
## [1] 19.96731
```

- We can guess (correctly!) that the expected value of a normal random variable matches its `mean` parameter.
- The expected value of a random variable is sometimes called its mean. We prefer "expected value" to distinguish from the "sample mean."

The expected value is not necessarily a possible value

Question 4.11. Find the expected value of a roll of a fair six-sided die.
(a) By using the definition $\mathrm{E}[X] = \sum_{i=1}^{\infty} x_i p_i$.

(b) By averaging a large number of simulated dice using R. Write some R code that is a starting point for testing and debugging.

The sample variance and the variance of a random variable

- The **sample variance** of $\mathbf{y} = (y_1, \ldots, y_n)$ is

$$\mathrm{var}(\mathbf{y}) = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- The **variance** of a random variable X is defined in terms of the expected value as

$$\mathrm{Var}(X) = \mathrm{E}\Big[\big(X - \mathrm{E}[X] \big)^2 \Big]$$

- If X is a random variable, then so is $Y = \big(X - \mathrm{E}[X] \big)^2$. Each possible outcome of X (say, $X = x$) matches an outcome $Y = \big(x - \mathrm{E}[X] \big)^2$.
- Collections of numbers have a *sample variance* computed by var (not capitalized). Random variables have a *variance* computed with Var (capitalized).
- People do not always make this distinction, but we will try to.
- In R, `var()` calculates the sample variance.

Standard deviation

- The **sample standard deviation** of $\mathbf{y} = (y_1, \ldots, y_n)$ is the square root of the sample variance.

$$\mathrm{sd}(\mathbf{y}) = \sqrt{\mathrm{var}(\mathbf{y})}$$

- The **standard deviation** of a random variable X is the square root of its variance.

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$$

- In R, `sd()` computes the sample standard deviation.
- We can compute $\mathrm{SD}(X)$ as the sample standard deviation of a large number of draws of the random variable X.

```
x <- rnorm(10000,mean=20,sd=5)
sd(x)
```

```
## [1] 5.061782
```

- As we might anticipate, this confirms that the `sd` parameter of the normal distribution matches its standard deviation.

Expectation, variance and standard deviation of $mX + c$

- Let X be a random variable and let $Y = mX + c$.
- Y is also a random variable. If X takes value x, Y takes value $mx + c$.
- Expectation is **linear**, meaning

$$\mathrm{E}[mX + c] = m\mathrm{E}[X] + c$$

- Variance doesn't depend on c. It is a measure of **spread**. Adding a constant shifts the center of a distribution but doesn't change the spread.
- Variance is quadratic in m.

$$\mathrm{Var}(mX + c) = m^2 \mathrm{Var}(X)$$

- Standard deviation therefore scales with m.

$$\mathrm{SD}(mX + c) = m\,\mathrm{SD}(X)$$

- SD scales nicely. Var can be easier to use for calculations.

The standard normal distribution

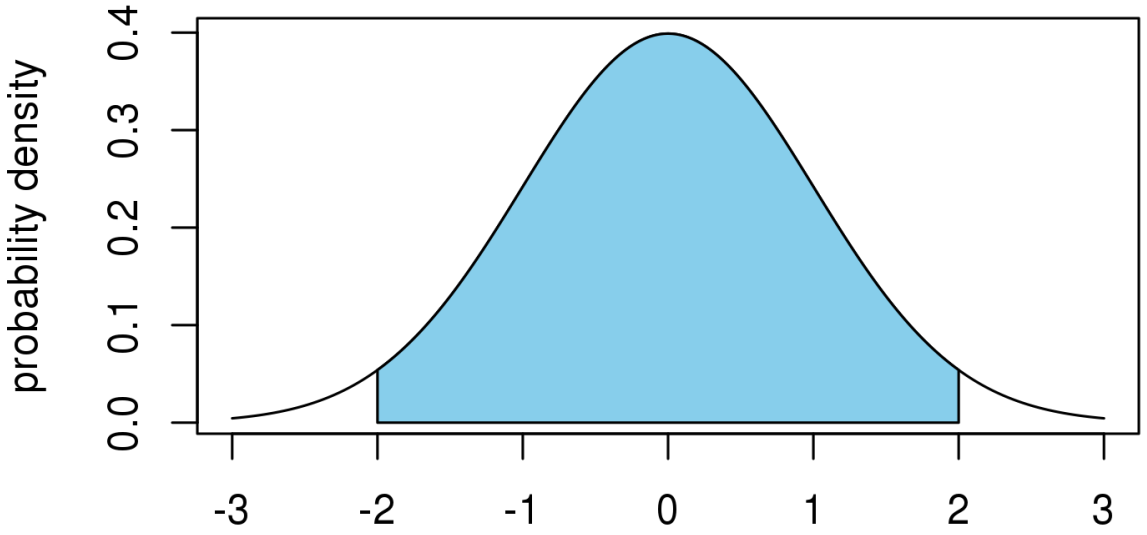
- Let $Z \sim \text{normal}(0, 1)$, so $\text{E}[Z] = 0$ and $\text{SD}(Z) = 1$.
- Z is called a **standard normal random variable**.
- Let $X = \mu + \sigma Z$.
- We use linearity of expectation and the scaling property of standard deviation to calculate

$$\text{E}[X] = \mu + \sigma\,\text{E}[Z] = \mu, \qquad \text{SD}(X) = \sigma\,\text{SD}(Z) = \sigma$$

- A bell curve is still a bell curve if you shift or rescale it, so X also follows a normal distribution.
- Therefore, $X \sim \text{normal}(\mu, \sigma)$.
- We can work the other way around: if $X \sim \text{normal}(\mu, \sigma)$ then $Z = (X - \mu)/\sigma$ has $Z \sim \text{normal}(0, 1)$.

Standardizing into standard units

- After subtracting the mean and dividing by the standard deviation, any normal random variable follows the standard normal distribution.
- This is called **standardizing**. We say we are working in **standard units**.
- Calculating in standard units was essential before computers: people used tables giving probabilities for the standard normal distribution.




```
pnorm(2)-pnorm(-2)

## [1] 0.9544997
```

- Thinking in standard units remains helpful. For example, as shown by the shaded area above, 95% of normal random variables are within 2 SD units of their mean.

Question 4.12. `Wikipedia:List_of_average_human_height_worldwide` says the average height of an American male aged 20-29 is 176.4 cm ($5'\,9.5''$). Suppose the standard deviation of height is $2.5''$. The average height of an NBA basketball player is about $6'\,7.5''$.

(a) Write $6'\,7.5''$ in standard units for this population.

(b) Estimate what percentage of male UM students are as tall as an average NBA player. Explain your assumptions. Sketch the answer as an area under the standard normal curve. Write this probability as an integral and show how to compute it via a call to `pnorm()`.