Midterm exam, STATS 401 W18

Name:

UMID:

Instructions. You have a time allowance of 80 minutes. The exam is closed book. Any electronic devices in your possession must be turned off and remain in a bag on the floor. If you need extra paper, please number the pages and put your name and UMID on each page.

Formulas

- You are not allowed to bring any notes into the exam.
- The following formulas will be provided. To use these formulas properly, you need to make appropriate definitions of the necessary quantities.

(1) $\mathbf{b} = \left(\mathbb{X}^\top \mathbb{X}\right)^{-1} \mathbb{X}^\top \mathbf{y}$

(2) $\mathrm{Var}(\hat{\beta}) = \sigma^2 \left(\mathbb{X}^\top \mathbb{X}\right)^{-1}$

(3) $\mathrm{Var}(\mathbb{A}\mathbf{Y}) = \mathbb{A}\mathrm{Var}(\mathbf{Y})\mathbb{A}^\top$

(4) $\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}[X^2] - \left(\mathrm{E}[X]\right)^2$

(5) $\mathrm{Cov}(X, Y) = \mathrm{E}\left[\left(X - \mathrm{E}[X]\right)\left(Y - \mathrm{E}[Y]\right)\right] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]$

(6) The binomial (n, p) distribution has mean np and variance $np(1-p)$.

From `?pnorm`:

```
pnorm(q, mean = 0, sd = 1)
qnorm(p, mean = 0, sd = 1)
q: vector of quantiles.
p: vector of probabilities.
```

Summation exercises

S1. A basic exercise.

Let $\mathbb{X} = [x_{ij}]$ be a 3×2 matrix with (i, j) entry given by $x_{ij} = 2i$.

(a) Write out $\mathbb{X}$, evaluating each of the six entries of the matrix.

(b) Hence, evaluate the sum $\sum_{i=1}^{3} \sum_{j=1}^{2} 2i$.

S2. An example involving the summation representation of matrix multiplication.

Evaluate $\mathbb{X}^{\mathrm{T}}\mathbb{X}$ where

$$\mathbb{X} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$$

R exercises

R1. Using `rep()` and `matrix()`.

Write the output of

```
matrix(c(rep(1,2), rep(0, 2), rep(0,2), rep(1,2)), nrow = 4)
```

R2. Manipulating vectors and matrices in R.

Which of the following is the output to `pnorm(c(-2,2))`

a) `[1] 0.02275013 0.97724987`

b) `Error in pnorm(c(-2,2)) : vector argument to scalar function`

c) `[1] 0.1586553 0.8413447`

d)
```
0.02275013
Warning message:
In pnorm(c(-2,2)) :
Vector argument to scalar function.
Function applied to only the first vector component.
```

e)
```
0.1586553
Warning message:
In pnorm(c(-2,2)) :
Vector argument to scalar function.
Function applied to only the first vector component.
```

Properties of variance and covariance

V1. A numerical calculation to find the variance of a linear combination using matrix techniques.

Let $\mathbf{X} = (X_1, X_2)$ be a vector random variable with mean $(3, 4)$ and variance matrix

$$\mathbb{V} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Let $Y = X_1 - X_2$. Find a suitable matrix $\mathbb{A}$ for which $Y = \mathbb{A}\mathbf{X}$, noting that the random variable Y can be considered as a 1×1 matrix. Set up and solve a matrix calculation to find the variance of Y.

V2. An algebraic calculation using basic definitions of variance & covariance, together with the linearity of expectation.

Use formulas (4) and (5) above, together with the linearity of expectation, to show that

$$\mathrm{Var}(3X + Y + 4) = 9\mathrm{Var}(X) + \mathrm{Var}(Y) + 6\mathrm{Cov}(X, Y)$$

.

Fitting a linear model by least squares

The director of admissions at a large state university wants to assess how well academic success can be predicted based on information available at admission. She collects a dataset of 705 students. She decides to take freshman GPA as the response variable, and she has access to ACT exam scores and percentile ranking of each student within their high school.

```
gpa <- read.table("gpa.txt",header=T)
```

```
head(gpa)
```

```
##    GPA High_School ACT
## 1 0.98          61  20
## 2 1.13          84  20
## 3 1.25          74  19
## 4 1.32          95  23
## 5 1.48          77  28
## 6 1.57          47  23
```

F1. Write the sample version of a linear model to address this question in subscript form.

F2. Write the sample version of this linear model in matrix form. Some of the quantities you have to define may be the same as the quantities you defined previously. Nevertheless, please make this model description self-contained.

F3. The following output fits a linear model in R.

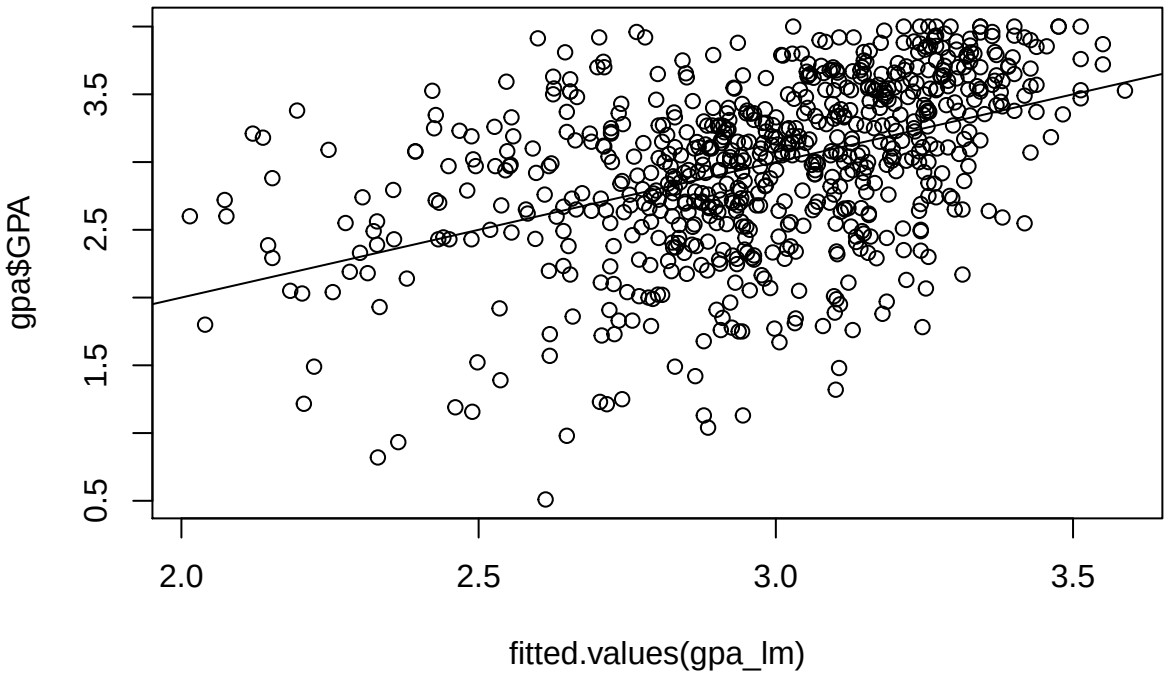
```
gpa_lm <- lm(GPA~ACT+High_School,data=gpa)
summary(gpa_lm)
```

```
##
## Call:
## lm(formula = GPA ~ ACT + High_School, data = gpa)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.10265 -0.29862  0.07311  0.40355  1.31336
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 1.292793   0.136725   9.455  < 2e-16 ***
## ACT         0.037210   0.005939   6.266 6.48e-10 ***
## High_School 0.010022   0.001279   7.835 1.74e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5672 on 702 degrees of freedom
## Multiple R-squared:  0.2033, Adjusted R-squared:  0.2011
## F-statistic: 89.59 on 2 and 702 DF,  p-value: < 2.2e-16
```

Explain how the coefficient estimates and the residual standard error presented in this output were calculated.

F4. Explain what the **fitted values** are for a linear model. Comment briefly on what the admissions director should learn (if anything) from the following plot of the freshman GPA of each patient plotted against the fitted value.

```
plot(x=fitted.values(gpa_lm),y=gpa$GPA)
abline(a=0,b=1)
```



The population version (or probability version) of the linear model

P1. Write out a suitable probability model, in subscript form, to give a population version of the linear model for freshman GPA in question F3. Some of the quantities you have to define may be the same as the quantities you defined previously. Nevertheless, please make this model description self-contained.

P2. Describe a suitable probability model, in matrix form, to give a population version of the linear model in question F3. Some of the quantities you have to define may be the same as the quantities you defined previously. Nevertheless, please make this model description self-contained.

P3. Explain how R produces standard errors for coefficients in a linear model. Also, describe in words how you interpret the standard error of 0.037210 for the coefficient of ACT.

Normal probability calculations

N1. A normal approximation to estimate a probability using the mean and variance.

Suppose that an analysis of a large dataset from another comparable university gave a coefficient of 0.03528 for the ACT variable when fitting a linear model using ACT score and high school rank. The admissions director is interested whether the difference could reasonably be chance variation due to having only a sample of 705 students, or whether the universities have differences beyond what can be explained by sample variation. Suppose that population value for this school is also 0.03528. Supposing the probability model in P1 and P2, and using a normal approximation, find an expression for the probability that the difference between the coefficient estimate for the data (0.03721) and the hypothetical true value (0.03528) is larger in magnitude than the observed value (0.03721-0.03528). Write your answer as a call to `pnorm()`. Your call to pnorm may involve specifying any necessary numerical calculations that you can't work out without access to a computer or calculator.

N2. A normal approximation to find a region with a given probability using the mean and variance.

Let $X_1, X_2, \ldots, X_n$ be independent random variables each of which take the value 0 with probability 0.5, 1 with probability 0.25 and -1 with probability 0.25. Find the mean and variance of X_1. Use this to find the mean and variance of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Now suppose $n = 100$ and suppose that $\bar{X}$ is well approximated by a normal distribution. Find a number c such that $\mathrm{P}(-c < \bar{X} < c)$ is approximately 0.9. Write your answer as a call to `qnorm()`. Your call to qnorm may involve specifying any necessary numerical calculations that you can't work out without access to a computer or calculator.
