Midterm 1

Math/Stats 425, Winter 2013
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1. While a driver is using a cell phone, he is three times more likely to get into an accident than if he were not using the phone. If he spends 1/8 of his time in the car using the phone, what is the chance he is using the phone when he has an accident?

   Solution: Let $E = \{$driver has an accident$\}$ and $F = \{$driver is using the phone$\}$, so $\mathbb{P}(E \mid F) = 3\mathbb{P}(E \mid F^c)$ and $\mathbb{P}(F) = 1/8$. From Bayes' formula,

   $$
   \begin{aligned}
   \mathbb{P}(F \mid E) &= \frac{\mathbb{P}(E \mid F)\mathbb{P}(F)}{\mathbb{P}(E \mid F)\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\mathbb{P}(F^c)} \\
   &= \frac{3\mathbb{P}(E \mid F^c) \times 1/8}{3\mathbb{P}(E \mid F^c) \times 1/8 + \mathbb{P}(E \mid F^c) \times 7/8} \\
   &= 3/10.
   \end{aligned}
   $$

2. A person has 8 friends, of whom 5 will be invited to a party. How many choices are there if 2 of the friends are feuding and must not both be invited?

   Solution: Label feuding friends A and B.

   $$
   \begin{aligned}
   \#\text{ ways} &= \#\{\text{ways with } A\} + \#\{\text{ways with } B\} + \#\{\text{ways without } A \text{ or } B\} \\
   &= \binom{6}{4} + \binom{6}{4} + \binom{6}{5} = 36.
   \end{aligned}
   $$

3. (a) What does it mean to say that E, F and G are independent events in a sample space $\mathbb{S}$?

   Solution: The definition is that $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$, $\mathbb{P}(E \cap G) = \mathbb{P}(E)\mathbb{P}(G)$, $\mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G)$ and $\mathbb{P}(E \cap F \cap G) = \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G)$.

   (b) Let E, F and G be three independent events in a sample space $\mathbb{S}$. Prove that E is independent of $F^c \cup G^c$.

   Solution: Writing $F^c \cup G^c = (F \cap G)^c$, we must check that $\mathbb{P}(E \cap [F \cap G]^c) = \mathbb{P}(E)\mathbb{P}([F \cap G]^c)$. Using the definition in part(t),

   $$
   \begin{aligned}
   \mathbb{P}(E \cap [F \cap G]^c) &= \mathbb{P}(E) - \mathbb{P}(E \cap [F \cap G]) \\
   &= \mathbb{P}(E) - \mathbb{P}(E)\mathbb{P}(F)\mathbb{P}(G) \\
   &= \mathbb{P}(E)\{1 - \mathbb{P}(F)\mathbb{P}(G)\} \\
   &= \mathbb{P}(E)\{1 - \mathbb{P}(F \cap G)\} \\
   &= \mathbb{P}(E)\mathbb{P}([F \cap G]^c)
   \end{aligned}
   $$

4. Anne and Bob decide to have children until they have both a boy and a girl. Supposing each is equally likely, and births are independent, find the chance that they end up with exactly 4 children.

<u>Solution</u>: Writing BG for the outcome of boy then girl, etc. According to A and B's decision, there are two ways to get exactly 4 kids:

$$\mathbb{P}(\text{exactly 4 kids}) = \mathbb{P}(\text{BGGG or GBBB}) = 2 \times \frac{1}{2^4} = \frac{1}{8}.$$

5. In a poker game, a 5 card hand is dealt from a shuffled deck of 52 cards. Any red cards (hearts or diamonds) are then discarded. An equal number of new cards are then dealt from the 47 remaining in the deck, replenishing the hand to 5 cards. Find an expression for the chance that the resulting poker hand consists only of black cards (spades or clubs).

   You are not asked to evaluate your expression. Your solution may be written in terms of a summation, such as $\sum_i p_i$ with some suitable expression for p_i and specified range of summation for the index i.

   <u>Solution</u>: Let $E = \{\text{final hand contains only black cards}\}$ and $F_i = \{\text{initial hand contains } i \text{ red cards}\}$ for $i = 0, \ldots, 5$. To count F_i, we can choose i red cards out of 26, followed by $5 - i$ black cards out of 26. Applying the basic principle of counting, together with equally likely outcomes among the $\binom{52}{5}$ possible hands, we obtain

   $$\mathbb{P}(F_i) = \binom{26}{i}\binom{26}{5-i} \bigg/ \binom{52}{5}. \tag{1}$$

   Given F_i, there are $26 - (5 - i)$ remaining black cards in the 47 cards available for the second draw. Therefore,

   $$\mathbb{P}(E \mid F_i) = \binom{21+i}{i} \bigg/ \binom{47}{i}. \tag{2}$$

   Putting (1) and (2) together, using the law of total probability,

   $$\mathbb{P}(E) = \sum_{i=0}^{5} \frac{\binom{26}{i}\binom{26}{5-i}\binom{21+i}{i}}{\binom{52}{5}\binom{47}{i}}.$$

   This gives an expression for the requested probability. Note that $\mathbb{P}(E \mid F_0) = 1$, which is formally consistent with (2) since $\binom{k}{0} = \frac{k!}{k!\,0!} = 1$ for all k.

6. Independent trials that result in a success with probability p are successively performed until a total of r successes is obtained. Show that the probability that exactly n trials are required is

   $$\binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

   <u>Solution</u>: This was in HW3. When "a total of r successes is obtained" we have $r-1$ successes before the last trial. So there is exactly $r-1$ successes between $n-1$ first trials and number of different ways is

   $$\binom{n-1}{r-1}$$

so the probability of having exactly $r-1$ successes between $n-1$ first trials is

$$\binom{n-1}{r-1}p^{r-1}(1-p)^{n-r}.$$

Trials are independent, also probability of success in the last trial is p , so the answer will be

$$\binom{n-1}{r-1}p^{r}(1-p)^{n-r}.$$