Midterm 2

Math/Stats 425, Winter 2013
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1. A salesperson travels from house to house, selling subscriptions to a magazine. At each house he makes a successful sale with probability 0.05. Suppose it take 2 minutes to visit each house. The salesperson continues working each day until he has sold 10 subscriptions. Find expressions for the mean, variance and standard deviation of the amount of time the salesperson works each day. You may use properties of standard distributions without proof. Comment on any assumptions your answer requires that are not explicitly given in the question.

   Solution: Let X be the number of houses visited and Y the amount of time worked. Supposing houses are independent, $X \sim \text{NegBinom}(r = 10, p = 0.05)$. This gives $\mathbb{E}[X] = r/p = 200$ and $\text{Var}(X) = r(1-p)/p^2 = 3800$. Then, $Y = 2X$ has $\mathbb{E}[Y] = 2\mathbb{E}[X] = 400$ minute, and $\text{Var}(Y) = 4\text{Var}(X) = 15200$ minute^2. Then, $\text{SD}(Y) = \sqrt{\text{Var}Y}$.

2. The average number of patients arriving at a large hospital with a broken limb is 5 per day if there is a snow fall, and 3 per day otherwise. During winter, the chance of snow falling on any particular day is 0.2. Suppose that no patients with broken limbs arrive on a particular day. What is the chance that snow fell on this day? Explain your assumptions.

   Solution: Let X be the number of broken limb patients, and let $E = \{\text{snow fell}\}$. Given E, the number of broken limb patients should be approximately conditionally distributed as Poisson(5), assuming that accidents are rare and approximately independent events. In particular, this assumption is reasonable only when accidents leading to multiple broken limb patients are especially rare. Similary, given E^c, $\mathbb{P}(X = k)$ should approximately match the Poisson(3) probability. So,

$$\begin{aligned}
\mathbb{P}(E \,|\, X = 0) &= \frac{\mathbb{P}(X = 0 \,|\, E)\mathbb{P}(E)}{\mathbb{P}(X = 0 \,|\, E)\mathbb{P}(E) + \mathbb{P}(X = 0 \,|\, E^{)}\mathbb{P}(E^c)} \\
&= \frac{0.2\, e^{-5}}{0.2\, e^{-5} + 0.8\, e^{-3}} = \frac{1}{1 + 4e^2}
\end{aligned}$$

3. Let $Y = X^2$, where X is a continuous random variable with density

$$f(x) = \begin{cases} (1/18)\sqrt{x} & 0 \le x \le 9 \\ 0 & x < 0 \text{ or } x > 9 \end{cases}$$

   (a) Find $\mathbb{E}[Y]$.

   Solution:

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^9 x^2 \frac{\sqrt{x}}{18}\, dx = \frac{1}{18}\left[\frac{2}{7}x^{7/2}\right]_0^9 \\
&= \frac{1}{63}\left\{3^7 - 0\right\} = \frac{3^5}{7} = 243/7.
\end{aligned}$$

   (b) Find the probability density function of Y. Hint: it may help you to first calculate the cumulative distribution function of Y.

Solution: Since $g(x) = \sqrt{(x)}$ is increasing on $x > 0$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(X \le \sqrt{y}) \\ &= \int_0^{\sqrt{y}} \frac{\sqrt{x}}{18}\, dx = \frac{1}{18}\left[\frac{2}{3}x^{3/2}\right]_0^{\sqrt{y}} \quad \text{for } 0 \le y \le 81. \end{aligned}$$

Now we differentiate to find

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{27}.\frac{3}{4}y^{-1/4} = \frac{1}{36y^{1/4}}\text{for } 0 \le y \le 81.$$

4. Let X be an exponential random variable, taking values on the positive real line with density

$$f(x) = \lambda\, e^{-\lambda x}.$$

Show that X has the memoryless property, that for any positive real numbers $a < b$,

$$\mathbb{P}(X > b \,|\, X > a) = \mathbb{P}(X > b - a).$$

Solution: The c.d.f. of X is $F(x) = \int_0^x f(y)\, dy = [-e^{-\lambda y}]_0^x = 1 - e^{-\lambda x}$. So, $\mathbb{P}(X > b) = 1 - F(x) = e^{-\lambda b}$. Then,

$$\begin{aligned} \mathbb{P}(X > b \,|\, X > a) &= \mathbb{P}(X > b, X > a)/\mathbb{P}(X > a) = \mathbb{P}(X > b)/\mathbb{P}(X > a) \\ &= \frac{[e^{-\lambda b}]}{e^{-\lambda a}} = e^{-\lambda(b-a)} \\ &= \mathbb{P}(X > b - a). \end{aligned}$$

5. A fair six-sided die is rolled three times. Let X be the smallest value of the three rolls, e.g. if the rolls are 2, 5 and 2 then X takes the value 2. Find an expression for the cumulative distribution function of X.

Solution: The c.d.f., for $k \in \{1, 2, \ldots, 6\}$, is

$$\begin{aligned} F_X(k) &= \mathbb{P}(X \le k) = 1 - \mathbb{P}(X \ge k + 1) \\ &= 1 - \mathbb{P}(\bigcap_{i=1}^{3} E_i) \quad \text{for } E_i = \{i\text{th roll is} \ge k + 1\} \\ &= 1 - [(6 - k)/6]^3. \end{aligned}$$

6. Let X have a Poisson(λ) distribution, with probability mass function $p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$, and let $Y = 2^{-X}$. Find the expected value of Y.

Solution: Using $\mathbb{E}[g(X)] = \sum_i g(x_i)\, p(x_i)$,

$$\begin{aligned} \mathbb{E}[Y] &= \sum_k 2^{-k}\frac{\lambda^k}{k!}e^{-\lambda} \\ &= e^{-\lambda}\sum_k \frac{(\lambda/2)^k}{k!} \\ &= e^{-\lambda}e^{\lambda/2} \\ &= e^{-\lambda/2} \end{aligned}$$