Sample final exam

Math/Stats 425 (Instructor: Edward Ionides)

1. Let X be a continuous random variable, with

$$P(X > x) = (1-x)^2, \qquad 0 \le x \le 1.$$

(i) Find the cumulative distribution function of X.

<u>Solution</u>: $F_X(x) = \mathbb{P}(X \le x) = 1 - (1-x)^2 = 2x - x^2$ for $0 \le x \le 1$.

(ii) Find the probability density function of X.

<u>Solution</u>: $f_X(x) = \frac{d}{dx} F_X(x) = 2(1-x)$ for $0 \le x \le 1$.

(iii) Find the expected value of X. <u>Solution</u>: $\mathbb{E}[X] = \int_0^1 x f_X(x)\, dx = \left[x^2 - 2x^3/3\right]_0^1 = 1/3$.

2. One evening, Fyodor decides to play 10 games of roulette, betting \$100 on black each time (this bet wins \$100 with probability $\frac{18}{38}$, and otherwise loses \$100).

(i) Find the expected value and standard deviation of his total winnings. You should write an expression which is suitable for evaluation, but you are not asked to evaluate it.

<u>Solution</u>: Let X be the number of games Fyodor wins, and Y his total winnings in dollars (negative if he makes a loss over all). Then, $X \sim \text{Binomial}(10, 18/38)$ and $Y = 100X - 100(10-X) = 200X - 1000$. So,

$$\mathbb{E}[X] = 10 \times \frac{18}{38}; \quad \text{Var}(X) = 10 \times \frac{18}{38} \times \frac{20}{38}$$

and

$$\begin{aligned}
\mathbb{E}[Y] &= 200\,\mathbb{E}[X] - 1000 = 2000 \times \frac{18}{38} - 1000 \\
\text{Var}(Y) &= 200^2\,\text{Var}(X) = 4 \times 10^5 \times \frac{18}{38} \times \frac{20}{38} \\
\text{SD}(Y) &= \sqrt{\text{Var}(Y)} = 100\sqrt{40 \times \frac{18}{38} \times \frac{20}{38}}
\end{aligned}$$

(ii) Find an expression for the exact chance (*i.e.*, not using a normal approximation) that he loses \$400 or more over the course of the evening.

<u>Solution</u>: If $X = 3$ then $Y = -400$, so

$$\begin{aligned}
\mathbb{P}(\text{loss is} \ge 400) &= \mathbb{P}(X \le 3) \\
&= \left(\frac{20}{38}\right)^1 0 + \binom{10}{1}\left(\frac{18}{38}\right)\left(\frac{20}{38}\right)^9 + \\
&\qquad \binom{10}{2}\left(\frac{18}{38}\right)^2\left(\frac{20}{38}\right)^8 + \binom{10}{3}\left(\frac{18}{38}\right)^3\left(\frac{20}{38}\right)^7.
\end{aligned}$$

3. Anne and Bin both plan to take the 16.10 train to Chicago. Anne's arrival at the train station is uniformly distributed between 0 and 20 minutes before the train departs. Bin's arrival is uniformly distributed between 0 and 15 minutes before the train departs. Find the chance that Anne arrives at the station before Bin. You may suppose that their arrivals are independent.

   Solution: Let X be the time that Anne arrives, in minutes before the train departs. Let Y be the time that Bin arrives, in minutes before the train departs. Note that $\{\text{A arrives before B}\} = \{X > Y\}$, and that the joint density $f(x,y)$ is $1/300$ on the rectangle $0 \le x \le 20$, $0 \le y \le 15$. Drawing the region C where this rectangle has $x > y$, we see that

$$\begin{aligned}\mathbb{P}(X > Y) &= \int\int_C f(x,y)\,dx\,dy \\ &= \frac{\text{Area of C}}{300} \\ &= 1 - \frac{0.5 \times 15 \times 15}{15 \times 20} = \frac{5}{8}.\end{aligned}$$

4. 24% of college freshmen and 16% of seniors regularly drink to excess. Find, approximately the chance that in a randomly selected sample of 200 freshmen and 200 seniors there are more freshmen who reguarly drink to excess. Write your answer in terms of the standard normal cumulative distribution function, $\Phi(x)$.

   Solution: Let X be the number of freshmen regularly drinking to excess, and Y the number of seniors. Assuming that the number of students is large, we can approximate $X \sim \text{Binomial}(200, 0.24)$ and $Y \sim \text{Binomial}(200, 0.16)$. Write

$$\begin{aligned}\mathbb{E}[X] &= \mu_X = 200 \times 0.24, \\ \text{Var}(X) &= \sigma_X^2 = 200 \times 0.24 \times 0.76, \\ \mathbb{E}[Y] &= \mu_Y = 200 \times 0.16, \\ \text{Var}(Y) &= \sigma_Y^2 = 200 \times 0.16 \times 0.84.\end{aligned}$$

   Let $\tilde{X} \sim \text{Normal}[\mu_X, \sigma_X^2]$ and $\tilde{Y} \sim \text{Normal}[\mu_Y, \sigma_Y^2]$ be two independent normal random variables approximating X and Y. Then,

$$\tilde{X} - \tilde{Y} \sim \text{Normal}[\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2].$$

   Now, using a continuity correction and taking $Z \sim \text{Normal}[0,1]$,

$$\begin{aligned}\mathbb{P}(X > Y) &= \mathbb{P}(X - Y > 0) \\ &\approx \mathbb{P}(\tilde{X} - \tilde{Y} > 1/2) \\ &= \mathbb{P}\left((\mu_X - \mu_Y) + \sqrt{\sigma_X^2 + \sigma_Y^2}\, Z > 1/2\right) \\ &= \mathbb{P}\left(Z > \frac{1/2 - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right) = 1 - \Phi\left(\frac{1/2 - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)\end{aligned}$$

5. A Math course has a sequence of three exams (Midterm 1, Midterm 2, Final). The chance of a student failing an exam is 1/20, unless that student failed at least one exam earlier in the course, in which case the chance of failing goes up to 1/3. Given that a student fails the final, find an expression for the chance that the student failed at least one of the two midterms.

   Solution: Let M_1, M_2 and F be the events that the student fails the first two midterms and final respectively. By Bayes' formula,

$$\mathbb{P}(M_1 \cup M_2 \mid F) = \frac{\mathbb{P}(F \mid M_1 \cup M_2)\mathbb{P}(M_1 \cup M_2)}{\mathbb{P}(F \mid M_1 \cup M_2)\mathbb{P}(M_1 \cup M_2) + \mathbb{P}\big[F \mid (M_1 \cup M_2)^c\big]\mathbb{P}\big[(M_1 \cup M_2)^c\big]}.$$

   And, using the law of total probability,

$$\begin{aligned} \mathbb{P}(M_1 \cup M_2) &= \mathbb{P}(M_1 \cup M_2 \mid M_1)\mathbb{P}(M_1) + \mathbb{P}(M_1 \cup M_2 \mid M_1^c)\mathbb{P}(M_1^c) \\ &= \frac{1}{20} + \frac{19}{20} \times \frac{1}{20} = \frac{39}{400}. \end{aligned}$$

   So,

$$\mathbb{P}(M_1 \cup M_2 \mid F) = \frac{\frac{1}{3} \times \frac{39}{400}}{\frac{1}{3} \times \frac{39}{400} + \frac{1}{20}\left(1 - \frac{39}{400}\right)}.$$

6. Suppose X has Poisson(λ) distribution, so $\mathbb{P}(X = k) = \lambda^k e^{-\lambda}/k!$ for $k = 0, 1, \ldots$. Suppose Y is independent of X and has Poisson(μ) distribution. Let $Z = X + Y$. Find the joint probability mass function of X and Z.

   Solution: The joint p.m.f. is $p(x, z) = \mathbb{P}(X = x, Z = z)$. X takes values in $\{0, 1, 2, \ldots\}$ and, given $X = x$, Z takes values in $\{x, x + 1, x + 2 \ldots\}$. Then,

$$\begin{aligned} p(x, z) &= \mathbb{P}(X = x, Y = z - x) \\ &= \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \quad \text{by independence} \\ &= \frac{\lambda^x e^{-\lambda}}{x!} \frac{\mu^{z-x} e^{-\mu}}{(z - x)!} \\ &= \frac{\lambda^x \mu^{z-x} e^{-(\lambda+\mu)}}{x!(z - x)!} \end{aligned}$$

   for $x = 0, 1, 2, \ldots$ and $z = x, x + 1, x + 2, \ldots$

7. Ten balls are dropped at random into two urns, A and B. Each ball independently falls into urn A with probability 1/2 and urn B with probability 1/2. Let X be the number of balls falling into urn A, and Y the number falling into B.

   (i) Find the variance of X.

   Solution: $X \sim \text{Binomial}(10, 1/2)$ so $\text{Var}(X) = 10 \times 1/2 \times 1/2 = 2.5$.

   (ii) Find the variance of $X + Y$.

   Solution: $X + Y = 10$, a constant, so $\text{Var}(X + Y) = 0$.

   (iii) Find the covariance of X and Y. Hint: it may help you to use parts (i) and (ii).

   Solution: $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$. Substituting in the aswers to (i) and (ii) gives $0 = 2.5 + 2.5 + 2\text{Cov}(X, Y)$ and so $\text{Cov}(X, Y) = -2.5$.