Sample Midterm 1

Math/Stats 425 (Instructor: Edward Ionides)

1. In a group of students, 70% are Democrat and 30% Republican. Of the Democrats, 60% support introduction of a ban on assault rifles. 30% of the Republicans are in favor of such a ban. If a randomly selected student opposes a ban, what is the chance that the student is Republican?

   Solution: Let $R = \{\text{student is Republican}\}$ and $S = \{\text{student supports a ban}\}$. From Bayes' rule,

$$\mathbb{P}(R \mid S^c) = \frac{\mathbb{P}(S^c \mid R)\mathbb{P}(R)}{\mathbb{P}(S^c \mid R)\mathbb{P}(R) + \mathbb{P}(S^c \mid R^c)\mathbb{P}(R^c)} = \frac{0.7 \times 0.3}{0.7 \times 0.3 + 0.4 \times 0.7} \tag{1}$$

2. A five card poker hand is dealt from a shuffled deck of 52 cards.

   (a) Find an expression for the chance that the hand contains a pair (i.e., two cards of the same rank, all other cards of different ranks).

   Solution: $\mathbb{S} = \{5 \text{ card poker hands}\}$, $E = \{\text{hand is worth a pair}\}$. Since outcomes are equally likely,

$$\mathbb{P}(E) = \frac{\#E}{\#\mathbb{S}} = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$$

   (b) Find an expression for the chance that the hand contains two pairs (i.e., two cards of one rank, two of another rank and a fifth card of yet another rank).

   Solution: Selecting ranks for the two pairs, followed by suits for each pair, followed by a rank and suit for the remaining card, we get

$$\mathbb{P}(\text{two pair}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}.$$

3. For three events E, F and G you know the following facts:

$$P(E) = 0.6, P(F) = 0.3, P(E \cap F) = 0.1, P(E \cap G) = 0.3, P(F \cap G) = 0.2.$$

   What are the largest and smallest possible values of $P(G)$? Hint: It may help you to set $x = P(E \cap F \cap G)$ and $y = P(G \cap E^c \cap F^c)$.

   Solution: Drawing a Venn diagram, using the properties that probabilities are non-negative and $\mathbb{P}(\mathbb{S}) = 1$, we obtain $0 \leq x \leq 0.1$ and $0 \leq y \leq 0.2$ as well as the expression

$$\mathbb{P}(G) = 0.5 + y - x,$$

   from which we deduce $0.4 \leq \mathbb{P}(G) \leq 0.7$.

4. The number of fish in a lake can be investigated by a capture-recapture experiment. Suppose that 10 fish are caught, tagged, and replaced in the lake. Next day, 20 more fish are caught.

Find an expression for the probability of recapturing k tagged fish if there are N fish in the lake. Comment on your assumptions.

Solution: Suppose the fish are labeled $1,\ldots,N$, with $1,\ldots,10$ being those who were tagged. Let $\mathbb{S}$ be unordered selections of 20 labels from $1,\ldots,N$. Assuming equally likely outcomes,

$$\mathbb{P}(k \text{ tagged fish caught}) = \frac{\binom{10}{k}\binom{N-10}{20-k}}{\binom{N}{20}}.$$

The equally likely outcome assumption is questionable here. Perhaps, the fish who got caught first time might be those which are easier to catch and therefore easier to recatch. Alternatively, those who are caught might be more careful next time!

5. Each morning, Xi works out by doing three sets of each of five exercises. A workout routine is a list giving the sequence of the sets, i.e. a list of fifteen sets in which each each exercise appears exactly three times. For example, labeling the exercises A, B, C, D, E a possible routine is $AABCABDBCEECDDE$. This concrete example should make clear what is meant by a routine for the purposes of this question.

   a) Find an expression for the number of possible routines Xi can choose.

   Solution: Assigning 15 positions to the labeled categories A, B, C, D, E,

   $$\# \text{ routines} = \binom{15}{3\ 3\ 3\ 3\ 3}.$$

   b) One of the exercises is press-ups. Xi does not like to do press-ups until she is warmed up, and so she would prefer if none of the first five sets were press-ups. How many routines satisfy this constraint?

   Solution: Assigning press-ups to 3 of 10 possible positions, and the remaining exercises to the remaining 12 positions,

   $$\# \text{ routines} = \binom{10}{3}\binom{12}{3\ 3\ 3\ 3}.$$

6. Andy and Ben alternate rolling a pair of dice, stopping either when Andy rolls a sum of 9 or when Ben rolls a sum of 7. Assuming that Andy rolls first, find the probability that he also makes the final roll.

   Solution: On any given turn, $\mathbb{P}(\text{A rolls sum of 9}) = 4/36$ and $\mathbb{P}(\text{B rolls sum of 7}) = 6/36$. Thus,

$$\begin{aligned}
\mathbb{P}(\text{A makes final roll}) &= \mathbb{P}\Big(\bigcup_{k=0}^{\infty}\{\text{A makes final roll on roll } 2k+1\}\Big) \\
&= \sum_{k=0}^{\infty}\left(\frac{32}{36}\right)^k\left(\frac{30}{36}\right)^k\frac{4}{36} \\
&= \frac{4}{36}\sum_{k=0}^{\infty}\left(\frac{8}{9}\frac{5}{6}\right)^k \\
&= \frac{1/9}{1-\frac{20}{27}}
\end{aligned}$$