Homework 1 (Math/Stats 425, Winter 2013)

Due Tuesday Jan 22, in class

1. A, B and C take turns flipping a coin. The first one to get a head wins. Define the sample space for this experiment as

$$\mathbb{S} = \begin{cases} 1, 01, 001, 0001, \ldots \\ 000\cdots \end{cases}$$

(a) Describe this sample space in words.

(b) Let $E = \{A \text{ wins}\}$ and $F = \{B \text{ wins}\}$. Define the following events formally, as subsets of the sample space $\mathbb{S}$:

(i) E.

(ii) F.

(iii) $(E \cup F)^c$.

<u>Solution</u>:

(a) In this experiment, A, B, and C take turns in flipping a coin. The game ends when someone flips a head. In this sample space, we denote the obtaining of a tail by "0" and denote the obtaining of a head by "1". And since three people flip a coin until one of them get a head, each possible event would be expressed by a string of 0s followed by a 1. And we can denote the event that nobody wins by a sequence of 0s. Therefore, the sample space would be

$$S = \begin{cases} 1, 01, 001, 0001, \cdots, \\ 0000\ldots \end{cases}$$

(b) Let's assume that A flips the coin first, then B, and then C, then A, and so forth.

- $E = \{A \text{ wins}\}$
  $E = \{1, 0001, 0000001, \cdots\} = \{\underbrace{000\cdots00}_{3n}1 : \ n = 0, 1, 2, \cdots\}$
- $F = \{B \text{ wins}\}$
  $F = \{01, 00001, 00000001, \cdots\} = \{\underbrace{000\cdots00}_{3n+1}1 : \ n = 0, 1, 2, \cdots\}$
- $(E \cup F)^c$
  $(E \cup F)^c = \{001, 000001, \cdots, 000\cdots\} = \{\underbrace{000\cdots00}_{3n+2}1 : \ n = 0, 1, 2, \cdots\} \cup \{0000\cdots\}$

2. A town with a population of 100,000 has three newspapers: I, II and III. The proportions of the population reading the papers are as follows:

I: 10 percent	I and II: 8 percent
II: 30 percent	I and III: 2 percent
III: 5 percent	II and III: 4 percent
I and II and III: 1 percent	

(a) Find the number of people who read only one newspaper.

(b) How many people read at least two newspapers?

(c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?

(d) How many people do not read any newspapers?

(e) How many people read only one morning paper and one evening paper?

Solution:

By using a Venn Diagram, we can get to know how many people are in each section. And it will give us the answers for (a) to (e).
(a) 20,000
(b) 12,000
(c) 11,000
(d) 68,000
(e) 10,000

3. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children and 1 has five children.

(a) If one of these families is chosen at random, what is the probability it has i children, for $i = 1, 2, \ldots, 5$?

(b) If one of the children is randomly chosen, what is the probability this child comes from a family having i children, for $i = 1, 2, \ldots, 5$?

(c) Give a brief verbal description of these results, explaining of the differences between the two sets of probabilities.

Solution:

(a) Let E_i denote the event that the chosen family has i children.
Then, since the total number of families is 20, the probability the chosen family has i children would be $P(E_i) = \frac{\#E_i}{20}$.
Thus, $P(E_1) = \frac{4}{20},\ P(E_2) = \frac{8}{20},\ P(E_3) = \frac{5}{20},\ P(E_4) = \frac{2}{20},\ P(E_5) = \frac{1}{20}$

(b) Let F_i denote the event that the chosen child comes from a family having i children.
Then, since the total number of children is $1{\cdot}4+2{\cdot}8+3{\cdot}5+4{\cdot}2+5{\cdot}1 = 48$, the probability the chosen child comes from a family having i children would be $P(F_i) = \frac{\#F_i}{48}$.
Thus, $P(F_1) = \frac{4}{48},\ P(F_2) = \frac{16}{48},\ P(F_3) = \frac{15}{48},\ P(F_4) = \frac{8}{48},\ P(F_5) = \frac{5}{48}$

4. If two dice are rolled, find the probability that the sum of their upturned faces equals i, for each $i = 2, 3, \ldots, 12$. Comment briefly on the assumptions you have made to carry out your computation.

Solution:

Let E_i denote the event that the sum of the upturned faces equals i.
Then the E_i 's $(i = 2, \cdots, 12)$ would be
$E_2 = \{(1,1)\}$
$E_3 = \{(1,2), (2,1)\}$

$E_4 = \{(1,3),(2,2),(3,1)\}$
$E_5 = \{(1,4),(2,3),(3,2),(4,1)\}$
$E_6 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
$E_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
$E_8 = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$
$E_9 = \{(3,6),(4,5),(5,4),(6,3)\}$
$E_{10} = \{(4,6),(5,5),(6,6)\}$
$E_{11} = \{(5,6),(6,5)\}$
$E_{12} = \{(6,6)\}$
Thus $P(E_2) = P(E_{12}) = \frac{1}{36},\ P(E_3) = P(E_{11}) = \frac{2}{36},\ P(E_4) = P(E_{10}) = \frac{3}{36},\ P(E_5) = P(E_9) = \frac{4}{36},\ P(E_6) = P(E_8) = \frac{5}{36},\ P(E_7) = \frac{6}{36}$

5. Let E and F be two events in a sample space $\mathbb{S}$, with $\mathbb{P}(E) = 0.9$ and $\mathbb{P}(F) = 0.8$.

   (i) Argue that $\mathbb{P}(E \cap F) \geq 0.7$.

   (ii) In general, prove Bonferroni's inequality, namely,

$$P(E \cap F) \geq P(E) + P(F) - 1.$$

   Instructions: Use a less formal approach for (i) than for (ii); otherwise, (i) becomes trivial as a special case of (ii). For (ii), a *proof* means an argument which explains the correctness of the result to your own satisfaction. A good proof will also explain the correctness of the result clearly to other people.

   Solution:

   Consider any two events E and F that are subsets of S, the sample space.
   It is known that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ by Prop. 4.3.
   And since $E \cup F \subseteq S$, we know that $0 \leq P(E \cup F) \leq P(S) = 1$
   Then we can get $1 \geq P(E) + P(F) - P(E \cap F)$
   Thus, $P(E \cap F) \geq P(E) + P(F) - 1$

   Especially when $P(E) = .9$ and $P(F) = .8$, $P(E \cap F) \geq .9 + .8 - 1 = .7$.