Homework 2 (Math/Stats 425, Winter 2013)

Due Tuesday Jan 29, in class

1. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

   Note: there is some ambiguity in this question. If you can see more than one way it can be interpreted, you should comment briefly on this and follow what you think is the most reasonable interpretation.

   Solution:

   One can use formula for permutations with indistinguishable objects, then the answer should be
$$\frac{12!}{6!4!1!1!} = 27720$$

2. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if

   (a) the books can be arranged in any order;

   (b) the mathematics books must be together and the novels must be together; (c) the novels must be together but the other books can be arranged in any order?

   Solution:

   (a) Since the total number of the books is 6, there are $6! = 720$ different arrangements.

   (b) 3 novels must be together and 2 math books must be together. If we treat 3 novels as one object and also treat 2 math books as one object, we will have 3 objects to be arranged including 1 chemistry book. So, we have $3! \times 3! \times 2! = 72$ different arrangements, since the orders of novels or math books can be switched within each group.

   (c) 3 novels must be together. If we treat 3 novels as one object, we will have 4 objects to be arranged including 2 math books and 1 chemistry book. Therefore, we have $4! \times 3! = 144$ different arrangements, since there exists 3! different orderings within the object of novels.

3. Poker dice is played by simultaneously rolling 5 dice. Show that

   (a) $\mathbb{P}\{\text{no two alike}\} = 0.0926$;

   (b) $\mathbb{P}\{\text{one pair}\} = 0.4630$;

   (c) $\mathbb{P}\{\text{two pairs}\} = 0.2315$;

   (d) $\mathbb{P}\{\text{three alike}\} = 0.1543$;

   (e) $\mathbb{P}\{\text{full house}\} = 0.0386$;

   (f) $\mathbb{P}\{\text{four alike}\} = 0.0193$;

(g) $\mathbb{P}\{\text{five alike}\} = 0.0008$;

Note: a full house is three of a kind together with a pair.

<u>Solution</u>:

Rolling 5 dice. The number of total events will be 6^5.

(a) We want to get 5 different numbers from 5 dice. Since there are $\binom{6}{5}$ ways to choose 5 different numbers from 1 to 6, and 5! different orderings for each of 5 dice, $\#(\textit{no two alike}) = \binom{6}{5} \times 5! = 720$. Therefore, the probability $P(\textit{no two alike}) = \frac{720}{6^5} = .0926$

(b) We can first choose which number will be a pair (from 1 to 6), and which dice will be a pair. So, there are $\binom{6}{1} \times \binom{5}{2}$ ways to choose a pair. And there are $\binom{5}{3} \times 3!$ possible different choices for the dice which is not a pair. So, $\#(\textit{one pair}) = \binom{6}{1} \times \binom{5}{2} \times \binom{5}{3} \times 3! = 3600$. And, therefore the probability $P(\textit{one pair}) = \frac{3600}{6^5} = .4630$.

(c) We can first choose which two numbers will be pairs, and which dice will be pairs, and then can choose one remaining number from 4 choices. So, $\#(\textit{two pairs}) = \binom{6}{2} \times \binom{5}{2} \times \binom{3}{2} \times \binom{4}{1} = 1800$ Therefore, the probability $P(\textit{two pairs}) = \frac{1800}{6^5} = .2315$.

(d) First we can choose which number will be the common number, and which dice will have the common number, and then can choose two remaining numbers from 5 choices. So, $\#(\textit{three alike}) = \binom{6}{1} \times \binom{5}{3} \times \binom{5}{2} \times 2! = 1200$ Therefore, the probability $P(\textit{three alike}) = \frac{1200}{6^5} = .1543$.

(e) Similarly, $\#(\textit{full house}) = \binom{6}{1} \times \binom{5}{3} \times \binom{5}{1} = 300$ Therefore, the probability $P(\textit{full house}) = \frac{300}{6^5} = .0386$.

(f) Similarly, $\#(\textit{four alike}) = \binom{6}{1} \times \binom{5}{4} \times \binom{5}{1} = 150$ Therefore, the probability $P(\textit{four alike}) = \frac{150}{6^5} = .0193$.

(g) Since we can choose which number will be the overall common number, $\#(\textit{five alike}) = \binom{6}{1} = 6$ Therefore, the probability $P(\textit{fivealike}) = \frac{6}{6^5} = .0008$.

4. A forest contains 20 elk, of which 5 are captured, tagged and then released. A certain time later 4 of the elk are captured. What is the probability that exactly 2 of these 4 have been tagged? Comment briefly on the assumptions you are making and their reasonableness (this kind of capture-recapture study, together with the assumptions required to do a probabilistic analysis of the data, is a fundamental procedure for ecological studies).

Solution:

The number of total events is $\binom{20}{4}$. And there are $\binom{5}{2} \times \binom{15}{2}$ different ways to choose 2 from the previously tagged elks, and 2 from the elks which have not been tagged before.

Therefore, the probability $P(2\ of\ 4\ captured\ elks\ have\ been\ tagged) = \dfrac{\binom{5}{2} \times \binom{15}{2}}{\binom{20}{4}}$.

To get this answer, we are assuming that each elk is equally likely to be chosen, and no elk was born or dead between the two capture points.

5. Five people, designated A, B, C, D, E, are arranged in a line. Assuming that each order is equally likely, find the chance that

   (a) there is exactly one person between A and B;

   (b) there are exactly two people between A and B;

   (c) there are three people between A and B.

   Solution:

   (a) If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $2 \times 3 \times 3!/5! = 3/10$

   (b) Similarly, this time if A is first, A can just have 2 choices. So the final answer is $2 \times 2 \times 3!/5! = 2/10$

   (c) $2 \times 1 \times 3!/5! = 1/10$

6. Given 20 people, what is the probability that among the 12 months of the year there are 4 months containing exactly 2 birthdays and 4 months containing exactly 3 birthdays? Comment briefly on your assumptions.

   Solution:

   One person's birthday can be in one of the 12 months, so the number of total events is 12^{20}. There are $\binom{12}{4} \times \binom{8}{4}$ ways to choose 4 months containing 2 birthdays and 4 containing 3 birthdays. Then divide 20 people as 8 groups of $(2,2,2,2,3,3,3,3)$.

   Probability= $\binom{12}{4}\binom{8}{4}\frac{(20)!}{(3!)^4(2!)^4}/12^{20}$