Homework 3 (Math/Stats 425, Winter 2013)

1. Suppose that 5 percent of men and 0.25 percent of women are colorblind. A colorblind person is chosen at random. What is the probability of this person being male?

   Solution:

   we want to get $\mathbb{P}(male|colorblind)$.

$$\mathbb{P}(male|colorblind) = \frac{\mathbb{P}(male \cap colorblind)}{\mathbb{P}(colorblind)} \tag{1}$$

$$= \frac{\mathbb{P}(colorblind|male)\mathbb{P}(male)}{\mathbb{P}(colorblind|male)\mathbb{P}(male) + \mathbb{P}(colorblind|female)\mathbb{P}(female)} \tag{2}$$

$$= \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.0025 \times 0.5} \tag{3}$$

$$= 0.95 \tag{4}$$

2. A red die, a blue die, and a yellow die (all six sided) are rolled. Let R, B, and Y be the respective numbers showing on the dice.

   (a) Find the probability no two dice land on the same number, i.e.,

   $$\mathbb{P}(\{R \neq B\} \cap \{R \neq Y\} \cap \{B \neq Y\}).$$

   (b) Given that no two dice land on the same number, find the conditional probability that the blue die shows less than the yellow die which shows less than the red die? i.e.,

   $$\mathbb{P}(B < Y < R \mid \{R \neq B\} \cap \{R \neq Y\} \cap \{B \neq Y\}).$$

   (c) Find $\mathbb{P}(B < Y < R)$.

   Solution:

   (a) Let E denote the event that no two of the dice land on the same number. Use the formula

   $$\mathbb{P}(E) = \frac{number\ of\ outcomes\ in E}{number\ of\ outcomes\ in\ S}$$

   So we have

   $$\mathbb{P}(E) = \frac{6 \times 5 \times 4}{6 \times 6 \times 6}$$

   (b) Given by E, we have 3! different (like$(B < Y < R)$ or $(B < R < Y)$,etc) outcomes and they are equally likely to occur. So

   $$\mathbb{P}(B < Y < R \mid E) = \frac{1}{3!} = \frac{1}{6}$$

   (c) Note that $\{B < Y < R\} \subset E$,

   $$\mathbb{P}(B < Y < R) = \mathbb{P}(E) \times \mathbb{P}(B < Y < R \mid E) = \frac{5}{9} \times \frac{1}{6} = \frac{5}{54}$$

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3. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed, or if one is blue-eyed and the other brown-eyed, then the person will have brown eyes. The brown-eyed gene is therefore said to be *dominant* over the blue-eyed gene. A newborn child independently receives one eye gene from each of its parents, chosen at random from the two eye genes belonging to each parent. Suppose that Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.

(a) What is the probability that Smith possesses a blue-eyed gene?

Now suppose that Smith's wife has blue eyes.

(b) What is the probability that their first child will have blue eyes?

(c) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

Solution:

Let B denote a brown-eyed gene and let b denote a blue-eyed gene. Then BB, Bb, bb will be the pairs of genes. The person who has BB *or* Bb will have brown eyes and the person who has bb will have blue eyes.

Let S be the pair of eye genes of Smith. Since the gene he receives from a parent is equally likely to be either of the two eye genes that the parent has, And since Smith's sister has blue eyes, we can conclude that the genes of Smith's parents are Bb's. Therefore, at birth, Smith was equally likely to receive either a blue gene or a brown gene from each parent.

$\mathbb{P}(\ \{S = BB\}\ ) = \frac{1}{4}, \quad \mathbb{P}(\ \{S = Bb\}\ ) = \frac{1}{2}, \quad prob(\ \{S = bb\}\ ) = \frac{1}{4}$

(a) Let A denote the event that Smith has one blue-eyed gene(that is, he has Bb) given that he has brown eyes.

$$\mathbb{P}(A) \quad = \quad \mathbb{P}(\ \{S = Bb\}\ |\ \{S = BB\ or\ S = Bb\}\ ) \tag{5}$$

$$= \quad \frac{\frac{1}{2}}{\frac{1}{4} + \frac{1}{2}} \tag{6}$$

$$= \quad \frac{2}{3} \tag{7}$$

(b) Let C_1 denote the eye genes of the first child from Smith and his wife. Then,

$$\mathbb{P}(\{C_1 = bb\}) \quad = \quad \mathbb{P}(\ \{C_1 = bb\}\ |\ A\ )\mathbb{P}(A) + \mathbb{P}(\ \{C_1 = bb\}\ |\ A^c\ )\mathbb{P}(A^c) \tag{8}$$

$$= \quad \frac{1}{2} \times \frac{2}{3} + \ 0 \tag{9}$$

$$= \quad \frac{1}{3} \tag{10}$$

where A^c means the event that Smith does not have any blue-eyed gene(that is, he has BB) given that he has brown eyes.

(c) Let $E = \{\text{Smith is Bb}\}$, $B_1 = \{\text{1st child is Bb}\}$ and $B_2 = \{\text{2nd child is Bb}\}$. Note that 1st child is Bb} is the same as 1st child has brown eyes}, in this context. Applying the

conditional law of total probability,

$$\begin{aligned} \mathbb{P}(B_2 \mid B_1) &= \mathbb{P}(B_2 \mid B_1, E)\mathbb{P}(E \mid B_1) + \mathbb{P}(B_2 \mid B_1, E^c)\mathbb{P}(E^c \mid B_1) \\ &= \mathbb{P}(B_2 \mid E)\mathbb{P}(E \mid B_1) + \mathbb{P}(B_2 \mid E^c)\mathbb{P}(E^c \mid B_1). \end{aligned} \tag{11}$$

Now apply Bayes' rule to give

$$\mathbb{P}(E \mid B_1) = \frac{\mathbb{P}(B_1 \mid E)\mathbb{P}(E)}{\mathbb{P}(B_1 \mid E)\mathbb{P}(E) + \mathbb{P}(B_1 \mid E^c)\mathbb{P}(E^c)} = \frac{1/2 \times 2/3}{1/2 \times 2/3 + 1 \times 1/3} = \frac{1}{2}.$$

Putting this into (11),

$$\mathbb{P}(B_2 \mid B_1) = 1/2 \times 1/2 + 1 \times 1/2 = 3/4.$$

4. If $A \subset B$, express the following probabilities as simply as possible:

(a) $\mathbb{P}(A \mid B)$;   (b) $\mathbb{P}(A \mid B^c)$;   (c) $\mathbb{P}(B \mid A)$;   (d) $\mathbb{P}(B \mid A^c)$.

Solution:

Suppose that $A \subset B$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A|B^c) = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(B^c)} = 0$$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1$$

$$\mathbb{P}(B|A^c) = \frac{\mathbb{P}(B \cap A^c)}{\mathbb{P}(A^c)} = \frac{\mathbb{P}(B) - \mathbb{P}(A)}{1 - \mathbb{P}(A)}$$

5. Independent trials that result in a success with probability p are successively performed until a total of r successes is obtained. Show that the probability that exactly n trials are required is

$$\binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

Solution:

When "a total of r successes is obtained" we have $r - 1$ successes before the last trial. So there is exactly $r - 1$ successes between $n - 1$ first trials and number of different ways is

$$\binom{n-1}{r-1}$$

so the probability of having exactly $r - 1$ successes between $n - 1$ first trials is

$$\binom{n-1}{r-1} p^{r-1} (1-p)^{n-r}.$$

Trials are independent, also probability of success in the last trial is p , so the answer will be

$$\binom{n-1}{r-1} p^r (1-p)^{n-r}.$$