Homework 5 (Math/Stats 425, Winter 2013)

Due Tuesday March 12, in class

1. A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. Suppose his daily demand is a binomial random variable with $n = 10$, $p = 1/3$.

   (a) How many papers should he purchase to maximize his expected profit?

   Hint: it is not enough to show that the newsboy is most likely to have 3 customers. The solution has to depend on the price that the newsboy buys and sells papers at. For example, make sure that your method would also work if the question is changed so that the boy buys papers at 5 cents and sells them at 15.

   (b) Comment briefly on the use of the binomial distribution as a probability model in this situation.

   <u>Solution</u>:

   The daily demand, D is $Binomial(10, \frac{1}{3})$. Let Y denote the number of papers sold during the day, and b denote the number of papers he purchased.
   Consider possible b's.

   (i) When b=2,

y	0	1	2
$profit$	-20	-5	10
$P(Y=y)$	0.017	0.087	0.896

   where, for $y < b, P(Y = y) = P(X = y) = \begin{pmatrix} 10 \\ y \end{pmatrix} (\frac{1}{3})^y (\frac{2}{3})^{10-y}$

   And, $E[profit] = (-20) \cdot 0.017 + (-5) \cdot 0.087 + 10 \cdot 0.896 = 8.185$

   (ii) When b=3,

y	0	1	2	3
$profit$	-30	-15	0	15
$P(Y=y)$	0.017	0.087	0.195	0.701

   And, $E[profit] = (-30) \cdot 0.017 + (-15) \cdot 0.087 + 0 \cdot 0.195 + 15 \cdot 0.701 = 8.700$

   (iii) When b=4,

y	0	1	2	3	4
$profit$	-40	-25	-10	5	20
$P(Y=y)$	0.017	0.087	0.195	0.260	0.441

   And, $E[profit] = (-40) \cdot 0.017 + (-25) \cdot 0.087 + (-10) \cdot 0.195 + 5 \cdot 0.260 + 20 \cdot 0.441 = 5.315$

   We have found a local maximum of expected profit at b=3. It is fairly clear this is also the global maximum.

2. A student is getting ready to take an important oral examination and is concerned about the possibility of having an 'on' day or an 'off' day. He figures that if he has an on day, then each

of his examiners will pass him independently of each other with probability 0.8, whereas, if he has an off day, this probability will be reduced to 0.4. Suppose that the student will pass the examination if a majority of the examiners pass him. If the student feels that he is twice as likely to have an on day as an off day, should he request an examination with 3 examiners or with 5 examiners?

Solution: Let A denote the event that the student has an "on" day, and let B denote the denote the event that he passes the examination. Then,

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

Since the student feels that he is twice as likely to have an on day as he is to have an off day, $P(A) = 2P(A^c)$, so combining this $P(A^c) + P(A) = 1$, we have $P(A) = \frac{2}{3},\ P(A^c) = \frac{1}{3}$.
With 3 examiners:

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= [\sum_{r=2}^{3} \begin{pmatrix} 3 \\ r \end{pmatrix} (0.8)^r (0.2)^{3-r}] \cdot \frac{2}{3} + [\sum_{r=2}^{3} \begin{pmatrix} 3 \\ r \end{pmatrix} (0.4)^r (0.6)^{3-r}] \cdot \frac{1}{3} \\ &= 0.715 \end{aligned}$$

With 5 examiners:

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= [\sum_{r=3}^{5} \begin{pmatrix} 5 \\ r \end{pmatrix} (0.8)^r (0.2)^{5-r}] \cdot \frac{2}{3} + [\sum_{r=3}^{5} \begin{pmatrix} 5 \\ r \end{pmatrix} (0.4)^r (0.6)^{5-r}] \cdot \frac{1}{3} \\ &= 0.734 \end{aligned}$$

Therefore, the student should choose 5 examiners.

3. A certain typing agency employs two typist. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second. You read an article carefully and find it has 6 errors. Estimate the chance that it was typed by the second typist. Explain your reasoning.

   Solution:

   Let X denote the number of errors that the first typist would make per article, and Y denote the number of errors that the second typist would make per article. Then both X and Y are poisson random variables with parameters 3 and 4.2 respectively.
   Let A the event that the article is typed by the second typist, and B denote the event that

the article has 6 errors. Then

$$\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\
&= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\
&= \frac{P(Y=6)\frac{1}{2}}{P(Y=6)\frac{1}{2} + P(X=6)\frac{1}{2}} \\
&= \frac{\frac{4.2^6}{6!}e^{-4.2}\frac{1}{2}}{\frac{4.2^6}{6!}e^{-4.2}\frac{1}{2} + \frac{3^6}{6!}e^{-3}\frac{1}{2}} \\
&= \frac{1}{1 + \frac{1}{1.4^6}e^{1.2}}
\end{aligned}$$

4. Let X be a binomial random variable with parameters n and p. Show that

$$\mathbb{E}\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

<u>Solution</u>:

Let X denote a binomial random variable with parameters n and p. Then

$$\begin{aligned}
E[\frac{1}{X+1}] &= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k+1)!} p^k (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \sum_{k=0}^{n} \frac{(n+1)!}{(n-k)!(k+1)!} p^{k+1} (1-p)^{n-k} \\
&= \frac{1}{(n+1)p} \left[\sum_{j=0}^{n+1} \frac{(n+1)!}{(n-j+1)!j!} p^j (1-p)^{n-j+1} - (1-p)^{n+1} \right] \\
&= \frac{1}{(n+1)p} \left[1 - (1-p)^{n+1} \right]
\end{aligned}$$

5. Let X be a binomial random variable with parameters n and p. What value of p maximizes $\mathbb{P}(X=k)$ for a given value of k in $\{0, 1, \ldots, n\}$?

<u>Solution</u>:

$$\begin{aligned}
P(X=k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
\frac{d}{dp} P(X=k) &= k \binom{n}{k} p^{k-1} (1-p)^{n-k} \\
&- (n-k) \binom{n}{k} p^k (1-p)^{n-k-1} = 0 \\
p &= \frac{k}{n}
\end{aligned}$$

6. Let X be a Poisson random variable with parameter λ. Find the value of λ which maximizes $\mathbb{P}(X = k)$ for a given non-negative integer k.

   Solution:

$$\begin{aligned}
P(X=k) &= \frac{\lambda^k}{k!} e^{-\lambda} \\
\frac{d}{d\lambda} P(X=k) &= k \frac{\lambda^{k-1}}{k!} e^{-\lambda} - \frac{\lambda^k}{k!} e^{-\lambda} = 0 \\
\lambda &= k
\end{aligned}$$

Recommended reading:
Sections 4.6–4.8 in Ross "A First Course in Probability," 8th edition. This course will not include the Zeta distribution (Sec. 4.8.4).