Homework 6 (Math/Stats 425, Winter 2013)

Due Tuesday March 19, in class

1. The lifetime in years of a laptop battery is a random variable having a probability density function given by
$$f(x) = c\,xe^{-x/2}$$
for $x \geq 0$, with c being a constant. Compute the expected lifetime of the battery.

2. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas tranins headed for destination B arrive at 15-minute intervals starting at 7:05 a.m.

   (a) If a passenger arrives at the station at a time uniformly distributed between 7 a.m. and 8 a.m. and then gets on the first train that arrives, what proportion of the time does this passenger go to destination A?

   (b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10?

3. Define a collection of events $\{E_a, 0 < a < 1\}$ having the property that $\mathbb{P}(E_a) = 1$ for all a but $\mathbb{P}\left(\bigcap_a E_a\right) = 0$. Explain why this could not happen for a finite or countably infinite collection of events.

   Hint: One way to proceed is to let X be uniform over (0,1) and define each E_a in terms of X.

4. Let $f(x)$ be the density of a normal random variable, i.e.,
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}. \tag{1}$$

   (a) Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $\frac{d^2}{dx^2} f(x) = 0$ when $x = \mu + \sigma$ or $x = \mu - \sigma$.

   (b) Sketch $f(x)$, showing the ordinate values at $x = \mu - \sigma$, $x = \mu$ and $x = \mu + \sigma$ and being careful to represent the property established in part (a).

5. Let X be a continuous random variable with density $f(x)$. In class we showed that, for a non-negative function $g(x)$,
$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx. \tag{2}$$
Prove the more general result, that equation (2) holds without requiring $g(x) \geq 0$. You may use the method that we used in class, but you should not use the result we established (i.e., equation (2) for $g(x) \geq 0$) without proof.

Recommended reading:

Sections 5.1–5.3 in Ross "A First Course in Probability," 8th edition. Question 4 concerns the normal distribution, but you do not have to know anything about this distribution other than the density in equation (1) to do this question.