Homework 6 (Math/Stats 425, Winter 2013)

Due Tuesday March 19, in class

1. The lifetime in years of a laptop battery is a random variable having a probability density function given by

$$f(x) = c\,xe^{-x/2}$$

for $x \geq 0$, with c being a constant. Compute the expected lifetime of the battery.

Solution:

Let X denote the lifetime of. Then

$$\begin{aligned}
\int_0^\infty xe^{-x/2}dx &= -2xe^{-x/2}\Big|_0^\infty + \int_0^\infty 2e^{-x/2}dx \\
&= 2 \\
c &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
2EX &= \int_0^\infty xxe^{-x/2}dx \\
&= -2x^2e^{-x/2}\Big|_0^\infty + \int_0^\infty 4xe^{-x/2}dx \\
&= 8 \\
EX &= 4
\end{aligned}$$

using integration by parts.

2. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas tranins headed for destination B arrive at 15-minute intervals starting at 7:05 a.m.

(a) If a passenger arrives at the station at a time uniformly distributed between 7 a.m. and 8 a.m. and then gets on the first train that arrives, what proportion of the time does this passenger go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10?

Solution:

(a) Let X denote the time at which the passenger arrives. $X \sim U(0, 60)$

$$\begin{aligned}
& P(goes\,to\,A) \\
= \;& P(5 < X < 15) + P(20 < X < 30) + P(35 < X < 45) + P(50 < X < 60) \\
= \;& \frac{40}{60} = \frac{2}{3}
\end{aligned}$$

(b) Let X denote the time at which the passenger arrives. $X \sim U(10, 70)$

$$\begin{aligned} & P(goes\,to\,A) \\ = \; & P(10 < X < 15) + P(20 < X < 30) + P(35 < X < 45) + P(50 < X < 60) + P(65 < X < 70) \\ = \; & \frac{40}{60} = \frac{2}{3} \end{aligned}$$

3. Define a collection of events $\{E_a, 0 < a < 1\}$ having the property that $\mathbb{P}(E_a) = 1$ for all a but $\mathbb{P}\left(\bigcap_a E_a\right) = 0$. Explain why this could not happen for a finite or countably infinite collection of events.

   Hint: One way to proceed is to let X be uniform over (0,1) and define each E_a in terms of X.

   Solution:

   Let $X \sim U(0,1)$ and $E_a = \{X \neq a\}$.
   Thus, $\forall a \in (0,1)$, $P(E_a) = P(X \neq a) = 1$. But $P(\bigcap_a E_a) = P(\emptyset) = 0$

4. Let $f(x)$ be the density of a normal random variable, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\}. \tag{1}$$

   (a) Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $\frac{d^2}{dx^2} f(x) = 0$ when $x = \mu + \sigma$ or $x = \mu - \sigma$.

   (b) Sketch $f(x)$, showing the ordinate values at $x = \mu - \sigma$, $x = \mu$ and $x = \mu + \sigma$ and being careful to represent the property established in part (a).

   Solution:

   Let $f(x)$ denote the density function of a normal random variable with mean μ and variance σ^2. Then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   Its first order derivative is

$$f'(x) = (-2 \times \frac{1}{2\sigma^2}(x-\mu)) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = -\frac{(x-\mu)}{\sqrt{2\pi}\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   And the second derivative is

$$f''(x) = -\frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (1 - \frac{(x-\mu)^2}{\sigma^2})$$

   So, we can easily check that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$.

5. Let X be a continuous random variable with density $f(x)$. In class we showed that, for a non-negative function $g(x)$,

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx. \tag{2}$$

   Prove the more general result, that equation (2) holds without requiring $g(x) \geq 0$. You may use the method that we used in class, but you should not use the result we established (i.e., equation (2) for $g(x) \geq 0$) without proof.

Solution:

$$\begin{aligned}\mathbb{E}[g(X)] &= \mathbb{E}[I_A g(X)] - \mathbb{E}[I_{A^c}(-g(X))] \\ &= \int_{-\infty}^{\infty} g(x)f(x)I_A(x)\,dx \\ &- \int_{-\infty}^{\infty} -g(x)f(x)I_{A^c}(x)\,dx. \\ &= \int_{-\infty}^{\infty} g(x)f(x)\,dx.\end{aligned}$$

where $A = \{x : g(x) \geq 0\}$, $A^c = \{x : g(x) < 0\}$, and I_A is the indicator fuuction of A, i.e. $I_A(x) = 1$ if x is in A and $I_A(x) = 0$ if x is not in A . Note that we used the result we proved in class, so you need to provide a proof for it, for example writing down the same proof we did in class.

Recommended reading:
Sections 5.1–5.3 in Ross "A First Course in Probability," 8th edition. Question 4 concerns the normal distribution, but you do not have to know anything about this distribution other than the density in equation (1) to do this question.