Homework 7 (Math/Stats 425, Winter 2013)

Due Tuesday April 9, in class

1. The annual rainfall (in inches) in a certain region is modeled as being normally distributed with $\mu = 40$ and $\sigma = 4$. According to this model, what is the probability that it will take over 10 years before a year occurs having rainfall above 50 inches? What assumptions are you making?

   Solution: Let X denote the annual rainfall, and E denote the event that it will take over 10 years starting from this year before a year occurs having a rain fall of over 50 inches. Then

$$\begin{aligned} P(X > 50) &= 1 - P(X \le 50) \\ &= 1 - \int_{-\infty}^{50} \frac{1}{\sqrt{2\pi}4} e^{-\frac{(x-40)^2}{32}} dx \\ &= 1 - \int_{-\infty}^{2.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ &= 1 - \Phi(2.5) \\ &= 1 - 0.9938. \end{aligned}$$

   Therefore,

$$P(E) = \{1 - (1 - 0.9938)\}^{10} = 0.9397$$

   We are assuming that the annual rainfall is independent from year to year.

2. The number of years a new radio functions is exponentially distributed with parameter $\lambda = 1/8$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years? Comment on your assumptions.

   Solution:

$$\begin{aligned} P(X > 8 + x | X > x) &= P(X > 8) \\ &= \int_8^{\infty} \lambda e^{-\lambda x} dx \\ &= e^{-1} \end{aligned}$$

3. Find the density function of $R = a\sin(\Theta)$, where a is a fixed constant and Θ is uniformly distributed on $(-\pi/2, \pi/2)$.

   Note: such a random variable R arises in the theory of ballistics. If a projectile is fired from the origin at an angle α from the earth with speed ν, then the point R at which it returns to the earth can be expressed as $R = (\nu^2/g)\sin(2\alpha)$ where g is the gravitational constant.

Solution: First, calculate cdf of R.

$$\begin{aligned} F_R(r) &= P\{Asin\Theta \leq r\} \\ &= P\{sin\Theta \leq \frac{r}{A}\} \\ &= P\{\Theta \leq arcsin(\frac{r}{A})\} \\ &= \frac{arcsin(\frac{r}{A}) + \frac{\pi}{2}}{\frac{\pi}{2} + \frac{\pi}{2}} \\ &= \frac{1}{\pi} arcsin(\frac{r}{A}) + \frac{1}{2} \end{aligned}$$

By differentiating cdf, we get pdf,

$$\begin{aligned} f_R(r) &= \frac{d}{dr}(\frac{1}{\pi} arcsin(\frac{r}{A}) + \frac{1}{2}) \\ &= \frac{1}{\pi} \frac{1}{\sqrt{1-(\frac{r}{A})^2}} \frac{1}{A} \\ &= \frac{1}{\pi\sqrt{A^2 - r^2}} \quad for - A \leq r \leq A \end{aligned}$$

4. Suppose that 3 balls are chosen successively, without replacement, from an urn containing 5 white balls and 8 red balls. Let X_i equal 1 if the ith ball drawn is white, and otherwise X_i equals 0. Write the joint probability mass function of

   (a) X_1 and X_2

   (b) X_1, X_2 and X_3

   (c) $X_1 + X_2$ and $X_1 + X_3$

   Solution:

   (a)

$$\begin{aligned} p(0,0) &= \frac{8}{13}\frac{7}{12} = \frac{14}{39} \\ p(0,1) &= \frac{8}{13}\frac{5}{12} = \frac{10}{39} \\ p(1,0) &= \frac{5}{13}\frac{8}{12} = \frac{10}{39} \\ p(1,1) &= \frac{5}{13}\frac{4}{12} = \frac{5}{39} \end{aligned}$$

   (b)

$$\begin{aligned} p(0,0,0) &= \frac{8}{13}\frac{7}{12}\frac{6}{11} = \frac{28}{143} \\ p(0,0,1) &= \frac{8}{13}\frac{7}{12}\frac{5}{11} = \frac{70}{429} \end{aligned}$$

Similarly,

$$
\begin{aligned}
p(0,1,0) &= p(1,0,0) = \frac{70}{429} \\
p(0,1,1) &= \frac{8}{13}\frac{5}{12}\frac{4}{11} = \frac{40}{429}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
p(1,0,1) &= p(1,1,0) = \frac{40}{429} \\
p(1,1,1) &= \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143}
\end{aligned}
$$

(c)

$$
\begin{aligned}
p(0,0) &= \frac{28}{143} \\
p(0,1) &= \frac{70}{429} \\
p(0,2) &= 0 \\
p(1,0) &= \frac{70}{429} \\
p(1,1) &= \frac{40}{429} + \frac{70}{429} \\
p(1,2) &= \frac{40}{429} \\
p(2,0) &= 0 \\
p(2,1) &= \frac{40}{429} \\
p(2,2) &= \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143}
\end{aligned}
$$

5. The joint probability density function of X and Y is given by

$$
f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad 0 < x < 1,\ 0 < y < 2
$$

(a) Verify that this is indeed a valid joint density function.

(b) Compute the marginal density function of X.

(c) Find $\mathbb{P}(X > Y)$.

(d) Find $\mathbb{P}(Y > 1/2 \,|\, X < 1/2)$.

(e) Find $\mathbb{E}[X]$.

(f) Find $\mathbb{E}[Y]$

<u>Solution</u>:

(a)

$$
\int_0^1 \int_0^2 \frac{6}{7}(x^2 + \frac{xy}{2}) dy dx = \int_0^1 \frac{6}{7}(2x^2 + x) dx = 1
$$

(b)

$$f_X(x) = \int_0^2 \frac{6}{7}(x^2 + \frac{xy}{2})dy = \frac{6}{7}(2x^2 + x)$$

(c)

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x \frac{6}{7}(x^2 + \frac{xy}{2})dydx \\ &= \frac{6}{7}\int_0^1 (x^2y + \frac{1}{4}xy^2) \mid_0^x dx \\ &= \frac{6}{7}\int_0^1 \frac{5}{4}x^3 dx = \frac{15}{56} \end{aligned}$$

(d)

$$P(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} f_X(x)dx = \int_0^{\frac{1}{2}} \frac{6}{7}(2x^2 + x)dx = \frac{6}{7}(\frac{2x^3}{3} + \frac{x^2}{2}) \mid_0^{\frac{1}{2}} = \frac{5}{28}$$

$$P(X < \frac{1}{2}, Y > \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^2 \frac{6}{7}(x^2 + \frac{xy}{2})dxdy = \frac{3}{56} + \frac{45}{28 \times 16}$$

By the definition of conditional probability,

$$\begin{aligned} P(Y > \frac{1}{2} \mid X < \frac{1}{2}) &= \frac{P(X < \frac{1}{2}, Y > \frac{1}{2})}{P(X > \frac{1}{2})} \\ &= \frac{3}{56} \times \frac{28}{5} + \frac{48}{28 \times 16} \times \frac{5}{28} \\ &= \frac{69}{80} = 0.8625 \end{aligned}$$

(e)

$$\begin{aligned} EX &= \int_0^1 x f_X(x)dx \\ &= \int_0^1 x(\frac{12}{7}x^2 + \frac{6}{7}x)dx \\ &= \frac{12}{7}\frac{x^4}{4} + \frac{6}{7}\frac{x^3}{3} \mid_0^1 \\ &= \frac{5}{7} \end{aligned}$$

(f)

$$\begin{aligned} EY &= \int_0^2 \int_0^1 y\frac{6}{7}(x^2 + \frac{xy}{2})dxdy \\ &= \frac{6}{7}\int_0^1 (2x^2 + \frac{4}{3}x)dx \\ &= \frac{6}{7}(\frac{2}{3}x^3 + \frac{2}{3}x^2) \mid_0^1 \\ &= \frac{8}{7} \end{aligned}$$

Recommended reading:
Sections 5.4, 5.5, 5.7 and 6.1 in Ross "A First Course in Probability," 8th edition. This course will not include the material in Section 5.6.