Homework 8 (Math/Stats 425, Winter 2013)

Due Tuesday April 16, in class

1. A man and a woman agree to meet at a certain location at about 12:30 p.m. Suppose the man arrives at a time uniformly distributed between 12:15 and 1:00, and the woman independently arrives at a time uniformly distributed between 12:00 and 12:45.

   (a) Find the probability that the first to arrive waits no longer than 5 minutes.

   (b) What is the probability that the man arrives first?

   Solution:

   Let X denote the minutes after 12pm when the man arrives, and Y denote the minutes after 12pm when the woman arrives. Then $X \sim Unif(15, 60), \quad Y \sim Unif(0, 45)$.

   (i) When the first to arrive waits no longer than 5 minutes, the difference between X and Y should be less than 5. So using a figure for the region of all possible (X, Y)s :

$$\begin{aligned} P(|Y - X| \leq 5) &= \frac{1}{45} \cdot \frac{1}{45} \left(\frac{10}{\sqrt{2}} \cdot 25\sqrt{2} + \frac{1}{2} \cdot 10^2 \right) \\ &= \frac{4}{27} \end{aligned}$$

   (ii) The man arriving first corresponds to X being smaller than Y. So

$$\begin{aligned} P(X < Y) &= \frac{1}{45} \cdot \frac{1}{45} \cdot \frac{1}{2} \cdot 30^2 \\ &= \frac{2}{9} \end{aligned}$$

2. Suppose that n points are independently chosen at random on the perimeter of a circle and we want the probability that they all lie in some semicircle (that is, we want the chance that there is a line passing through the center of the circle such that all the points are on one side of the line).

   Let $P_1, \ldots, P_n$ denote the positions of the n points. Let A be the event that all n points lie in some semicircle. Let A_i be the event that all points lie in the semicircle beginning at the point P_i and going clockwise for $180°$, for $i = 1, \ldots, n$.

   (a) Express A in terms of $A_1, \ldots, A_n$.

   (b) Are $A_1, \ldots, A_n$ mutually exclusive?

   (c) Find $\mathbb{P}(A)$.

   Solution:

   (a) $A = \cup_{i=1}^{n} A_i$

   (b) Yes, they are disjoint.

   (c) $P(A) = \sum_{i=1}^{n} P(A_i) = n(\frac{1}{2})^{n-1}$

3. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} x+y & 0<x<1,\, 0<y<1 \\ 0 & \text{else} \end{cases}$$

(a) Are X and Y independent?

(b) Find the marginal density function of X.

(c) Find $\mathbb{P}(X+Y<1)$.

Solution:

(a,b)

$$\begin{aligned} f_X(x) &= \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy \\ &= x + \frac{1}{2} \end{aligned}$$

$$f_Y(y) = y + \frac{1}{2}$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, they are not independent.

(c)

$$\begin{aligned} P(X+Y<1) &= \int_0^1 \int_0^{1-x} f(x,y)dydx \\ &= \int_0^1 \int_0^{1-x} (x+y)dydx \\ &= \int_0^1 (x(1-x) + \frac{(1-x)^2}{2})dx \\ &= \int_0^1 (-\frac{x^2}{2} + \frac{1}{2})dydx \\ &= \frac{1}{3} \end{aligned}$$

4. Suppose X_1 and X_2 are independent exponential random variables with parameters λ_1 and λ_2 respectively.

(a) Obtain the density of $Z = X_1/X_2$.

(b) Compute $\mathbb{P}(X_1 < X_2)$.

Solution: Since X_1 and X_2 are independent, the joint density of X_1 and X_2 is

$$f_{X_1,X_2}(x,y) = \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y}, \quad x,y>0$$

Let $Z = \frac{X_1}{X_2}$. Then the distribution of Z is

$$\begin{aligned}
F_Z(z) &= P(Z \le z) \\
&= P(\frac{X_1}{X_2} \le z) \\
&= P(X_1 \le zX_2) \\
&= \int_0^\infty \int_0^{zy} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dx dy \\
&= \int_0^\infty (1 - e^{-\lambda_1 z y}) \lambda_2 e^{-\lambda_2 y} dy \\
&= 1 - \frac{\lambda_2}{\lambda_1 z + \lambda_2} \\
&= \frac{\lambda_1 z}{\lambda_1 z + \lambda_2}
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz} \frac{\lambda_1 z}{\lambda_1 z + \lambda_2} \\
&= \frac{\lambda_1 \lambda_2}{(\lambda_1 z + \lambda_2)^2}
\end{aligned}$$

And also

$$\begin{aligned}
P(X_1 < X_2) &= P(\frac{X_1}{X_2} < 1) \\
&= P(Z < 1) \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

5. Jill's bowling scores are approximately normally distributed with mean 170 and standard deviation 20, whereas Jack's scores are approximately normally distributed with mean 160 and standard deviation 15. If Jack and Jill each bowl one game, approximate the probability that

   (a) Jack's score is higher;

   (b) the total of their scores is above 350.

   Explain any assumptions you may require that are not explicitly specified in the question.

   <u>Solution</u>:

   Let X denote Jill's score and let Y denote Jack's score. Also, let Z be a standard normal random variable.

   (a)

$$\begin{aligned}
P\{Y > X\} &= P\{Y - X > 0\} \\
&= P\left\{ \frac{Y - X - (160 - 170)}{\sqrt{20^2 + 15^2}} > \frac{10}{\sqrt{20^2 + 15^2}} \right\} \\
&= P\{Z > 0.4\} \approx 0.3372
\end{aligned}$$

(b)

$$\begin{aligned} P\{X+Y>350\} &= P\left\{\frac{X+Y-330}{\sqrt{20^2+15^2}} > \frac{20}{\sqrt{20^2+15^2}}\right\} \\ &= P\{Z>0.8\} \approx 0.2061 \end{aligned}$$

6. Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed continuous random variables, each with density f. Let Z be the largest value taken by any of these random variables,

$$Z = \max\{X_1, X_2, \ldots, X_n\}.$$

Find the probability density function of Z.

Solution:

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(X_1 \le z)^n \\ &= F(z)^n \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= n f(z) F(z)^{n-1} \end{aligned}$$

Recommended reading:
Sections 6.2, 6.3, 6.4, 6.5 in Ross "A First Course in Probability," 8th edition.