Homework 9 (Math/Stats 425, Winter 2013)

Due Tuesday April 23, in class

1. The joint probability mass function of X and Y is given by

$$\begin{aligned} p(1,1) &= 1/8 \qquad & p(1,2) &= 1/4 \\ p(2,1) &= 1/8 \qquad & p(2,2) &= 1/2 \end{aligned}$$

   (a) Compute the conditional mass function of X given $Y = i$ for $i \in \{1, 2\}$.

   (b) Are X and Y independent?

   (c) Compute $\mathbb{P}(XY < 3)$, $\mathbb{P}(X + Y > 2)$, and $\mathbb{P}(X/Y > 1)$.

   Solution:

   (a)

$$\begin{aligned} P(Y = 1) &= p(1,1) + p(2,1) = \frac{1}{4} \\ P(Y = 2) &= p(1,2) + p(2,2) = \frac{3}{4} \end{aligned}$$

   Therefore,

$$\begin{aligned} P(X = 1|Y = 1) &= \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2} \\ P(X = 2|Y = 1) &= \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2} \\ P(X = 1|Y = 2) &= \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \\ P(X = 2|Y = 2) &= \frac{P(X = 2, Y = 2)}{P(Y = 2)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \end{aligned}$$

   (b) No, because $P(X = 1|Y = 1) = \frac{1}{2}$, $P(X = 1) = p(1,1) + p(1,2) = \frac{3}{8}$, and so $P(X = 1|Y = 1) \neq P(X = 1)$.

   (c)

$$\begin{aligned} P(XY \leq 3) &= p(1,1) + p(1,2) + p(2,1) = \frac{1}{2} \\ P(X + Y > 2) &= p(1,2) + p(2,1) + p(2,2) = \frac{7}{8} \\ P(X/Y > 1) &= p(2,1) = \frac{1}{8} \end{aligned}$$

2. The joint density function of X and Y is given by

$$f(x,y) = x\,e^{-x(y+1)}, \quad x > 0,\ y > 0.$$

Find the conditional density of X given $Y = y$ and the conditional density of Y given $X = x$.

Solution:

(a)

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dx} = (y+1)^2 xe^{-x(y+1)},\ x > 0$$

(b)

$$f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}dy} = xe^{-xy},\ y > 0$$

3. N people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then either sits at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the $\binom{N}{2}$ pairs of people are, independently, friends with probability p, find the expected number of occupied tables.

Hint: you might want to consider indicator random variables for the event that the ith arrival sits at a previously unoccupied table.

Solution:

Let X_i be the indicator of whether i^{th} arrival sits at a previously unoccupied table, that is, X_i is one when i^{th} arrival sits at unoccupied table and otherwise zero. Then

$$E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{N} E[X_i]$$

And,

$$\begin{aligned} E[X_i] &= P\{i^{th}\ arrival\ is\ not friends\ with\ any\ of\ first\ i-1\} \\ &= (1-p)^{i-1}. \end{aligned}$$

Therefore,

$$E[number\ of\ occupied\ tables] = \sum_{i=1}^{N} (1-p)^{i-1}$$

4. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n. Each ball is placed independently, with ball i equally likely to go into any of the urns numbered $1, 2, \ldots, i$. Find

(a) the expected number of urns that are empty;

(b) the probability that none of the urns is empty.

Solution:

Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$$E[X_j] = P\{ball\ i\ is\ not\ in\ urn\ j,\ i \geq j\} = \prod_{i=j}^{n}(1 - 1/i)$$

Hence,

(a)

$$E[number\ of\ empty\ urns] = \sum_{j=1}^{n}\prod_{i=j}^{n}(1 - 1/i)$$

(b)

$$P(none\ are\ empty) = P(ball\ j\ is\ in\ urn\ j,\ for\ all\ j) = \prod_{j=1}^{n} 1/j$$

5. If X and Y are independent and identically distributed with mean μ and variance σ^2, find

$$\mathbb{E}\big[(X - Y)^2\big].$$

Solution:

$$\begin{aligned} E[(X-Y)^2] &= E[X^2 - 2XY + Y^2] \\ &= E(X^2) + E(Y^2) - 2E(X)E(Y) \\ &= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 \\ &= Var(X) + Var(Y) = 2\sigma^2 \end{aligned}$$

6. A group of 20 people—consisting of 10 men and 10 women—are randomly arranged into 10 pairs of 2 each.

(a) Compute the expectation and the variance of the number of pairs that consist of a man and a woman.

(b) Now suppose the 20 people consisted of 10 married couples. Compute the mean and variance of the number of married couples that are paired together.

Solution:

(a) Let $X_i = \begin{cases} 1, & pair\ i\ consists\ of\ a\ man\ and\ a\ woman; \\ 0, & otherwise \end{cases}$

Then

$$\begin{aligned} E[X_i] &= P(X_i = 1) = \frac{10}{19} \\ E[X_iX_j] &= P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1) = \frac{10}{19}\frac{9}{17},\ i \neq j \\ E\left[\sum_{i=1}^{10} X_i\right] &= \frac{100}{19} \\ Var\left(\sum_{i=1}^{10} X_i\right) &= 10\frac{10}{19}(1 - \frac{10}{19}) + 10 \cdot 9\left[\frac{10}{19}\frac{9}{17} - \left(\frac{10}{19}\right)^2\right] = \frac{900}{19^2}\frac{18}{17} \end{aligned}$$

(b) Let $X_i = \begin{cases} 1, & pair\ i\ consists\ of\ a\ married\ couple; \\ 0, & otherwise \end{cases}$

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\begin{aligned}
E[X_i] &= P(X_i = 1) = \frac{1}{19} \\
E[X_i X_j] &= P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1 | X_i = 1) = \frac{1}{19}\frac{1}{17}, \ i \neq j \\
E\left[\sum_{i=1}^{10} X_i\right] &= \frac{10}{19} \\
Var\left(\sum_{i=1}^{10} X_i\right) &= 10\frac{1}{19}(1 - \frac{1}{19}) + 10 \cdot 9 \left[\frac{1}{19}\frac{1}{17} - \left(\frac{1}{19}\right)^2\right] = \frac{180}{19^2}\frac{18}{17}
\end{aligned}
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Recommended reading:
Sections 7.1, 7.2, 7.3, 7.4 in Ross "A First Course in Probability," 8th edition.