3. Conditional probability & independence

Conditional Probabilities

- **Question**: How should we modify $\mathbb{P}(E)$ if we learn that event F has occurred?

- **Derivation**: Suppose we repeat the experiment n times. Let $n(E \cap F)$ be the number of times that both E and F occur, and $n(F)$ the number of times F occurs.

- The proportion of times E occurs only counting trials where F occurs is

$$\frac{n(E \cap F)}{n(F)} = \frac{n(E \cap F)/n}{n(F)/n} \approx \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

- **Definition**: the conditional probability of E given F is

$$\boxed{\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}, \quad \text{for} \quad \mathbb{P}(F) > 0}$$

Example 1. 27 students out of a class of 43 are engineers. 20 of the students are female, of whom 7 are engineers. Find the probability that a randomly selected student is an engineer given that she is female.

Example 2. Deal a 5 card poker hand, and let

$E = \{\text{at least 2 aces}\}, \quad F = \{\text{at least 1 ace}\},$

$G = \{\text{hand contains ace of spades}\}.$

(a) Find $\mathbb{P}(E)$

(b) Find $\mathbb{P}(E \mid F)$

(c) Find $\mathbb{P}(E \mid G)$

The Multiplication Rule

- Re-arranging the conditional probability formula gives

  $$\boxed{\mathbb{P}(E \cap F) = \mathbb{P}(F)\,\mathbb{P}(E \mid F)}$$

  This is often useful in computing the probability of the intersection of events.

Example. Draw 2 balls at random without replacement from an urn with 8 red balls and 4 white balls. Find the chance that both are red.

The General Multiplication Rule

$$\boxed{\begin{aligned}&\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \\ &\qquad \mathbb{P}(E_1) \times \mathbb{P}(E_2 \,|\, E_1) \times \mathbb{P}(E_3 \,|\, E_1 \cap E_2) \times \\ &\qquad\qquad \cdots \times \mathbb{P}(E_n \,|\, E_1 \cap E_2 \cap \cdots \cap E_{n-1})\end{aligned}}$$

Example 1. Anil and Beth roll two dice, and play a game as follows. If the total is 5, A wins. If the total is 7, B wins. Otherwise, they play a second round, and so on. Find $\mathbb{P}(E_n)$, for $E_n = \{\text{A wins on } n\text{th round}\}$.

Example 2. I have n keys, one of which opens a lock. Trying keys at random without replacement, find the chance that the kth try opens the lock.

The Law of Total Probability

- From axiom A3, $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$. Using the definition of conditional probability,

  $$\boxed{\mathbb{P}(E) = \mathbb{P}(E \,|\, F)\,\mathbb{P}(F) + \mathbb{P}(E \,|\, F^c)\,\mathbb{P}(F^c)}$$

- This is **extremely useful**. It may be difficult to compute $\mathbb{P}(E)$ directly, but easy to compute it once we know whether or not F has occurred.

- To generalize, say events $F_1, \ldots, F_n$ form a **partition** if they are disjoint and $\bigcup_{i=1}^n F_i = \mathbb{S}$.

- Use a Venn diagram to argue that $\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E \cap F_i)$.

- Apply conditional probability to give the **law of total probability**,

  $$\boxed{\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E \,|\, F_i)\,\mathbb{P}(F_i)}$$

Example 1. Eric's girlfriend comes round on a given evening with probability 0.4. If she does not come round, the chance Eric watches *The Wire* is 0.8. If she does, this chance drops to 0.3. Find the probability that Eric gets to watch *The Wire*.

Bayes Formula

- Sometimes $\mathbb{P}(E \mid F)$ may be specified and we would like to find $\mathbb{P}(F \mid E)$.

Example 2. I call Eric and he says he is watching *The Wire*. What is the chance his girlfriend is around?

- A simple manipulation gives **Bayes' formula**,

$$\boxed{\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\,\mathbb{P}(F)}{\mathbb{P}(E)}}$$

Proof.

- Combining this with the law of total probability,

$$\boxed{\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\,\mathbb{P}(F)}{\mathbb{P}(E \mid F)\,\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\,\mathbb{P}(F^c)}}$$

Solution to Example 2.

- this computation can be viewed using a tree:

- Sometimes conditional probability calculations can give quite unintuitive results.

Example 3. I have three cards. One is red on both sides, another is red on one side and black on the other, the third is black on both sides. I shuffle the cards and put one on the table, so you can see that the upper side is red. What is the chance that the other side is black?

- is it $1/2$, or $> 1/2$ or $< 1/2$?

Solution

Discussion problem. Suppose 99% of people with HIV test positive, 95% of people without HIV test negative, and 0.1% of people have HIV. What is the chance that someone testing positive has HIV?

Example: Statistical inference via Bayes' formula

Rosencrantz and Guildenstern play a game where R tosses a coin, and wins \$1 if it lands on H or loses \$1 on T. G is surprised to find that he loses the first ten times they play. If G's **prior belief** is that the chance of R having a two headed coin is 0.01, what is his **posterior belief**?

Note. Prior and posterior beliefs are assessments of probability before and after seeing an outcome. The outcome is called **data**.

Solution.

Independence

- Heuristically, E is independent of F if the chance of E occurring is not affected by whether F occurs, i.e.,

$$\mathbb{P}(E \mid F) = \mathbb{P}(E) \tag{1}$$

- We say that E and F are **independent** if

$$\boxed{\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)} \tag{2}$$

Note. (2) is a rearrangement of (1). Check this!

Note. It is clear from (2) that independence is a symmetric relationship. Also, (2) is properly defined when $\mathbb{P}(F) = 0$.

Note. (1) is a useful way to think about independence; (2) is usually better to do the math.

Proposition. If E and F are independent, then so are E and F^c.

Proof.

Example 1: Independence can be obvious

Draw a card from a shuffled deck of 52 cards. Let E = card is a spade and F = card is an ace. Are E and F independent?

Solution

Example 2: Independence can be surprising

Toss a coin 3 times. Define

A={at most one T}=$\{HHH, HHT, HTH, THH\}$

B={both H and T occur}=$\{HHH, TTT\}^c$.

Are A and B independent?

Solution

Independence as an Assumption

- It is often convenient to suppose independence. People sometimes assume it without noticing.

Example. A sky diver has two chutes. Let

$$E = \{\text{main chute opens}\}, \qquad \mathbb{P}(E) = 0.98;$$

$$F = \{\text{backup opens}\}, \qquad \mathbb{P}(F) = 0.90.$$

Find the chance that at least one opens, **making any necessary assumption clear**.

Note. Assuming independence does not justify the assumption! Both chutes could fail because of the same rare event, such as freezing rain.

Independence of Several Events

- Three events E, F, G are **independent** if

$$\begin{aligned}\mathbb{P}(E \cap F) &= \mathbb{P}(E) \cdot \mathbb{P}(F) \\ \mathbb{P}(F \cap G) &= \mathbb{P}(F) \cdot \mathbb{P}(G) \\ \mathbb{P}(E \cap G) &= \mathbb{P}(E) \cdot \mathbb{P}(G) \\ \mathbb{P}(E \cap F \cap G) &= \mathbb{P}(E) \cdot \mathbb{P}(F) \cdot \mathbb{P}(G)\end{aligned}$$

- If E, F, G are independent, then E will be independent of any event formed from F and G.

Example. Show that E is independent of $F \cup G$.

Proof.

Pairwise Independence

- E, F and G are **pairwise independent** if E is independent of F, F is independent of G, and E is independent of G.

Example. Toss a coin twice. Set $E=\{HH,HT\}$, $F=\{TH,HH\}$ and $G=\{HH,TT\}$.

(a) Show that E, F and G are pairwise independent.

(b) By considering $\mathbb{P}(E\cap F\cap G)$, show that E, F and G are NOT independent.

Note. Another way to see the dependence is that $\mathbb{P}(E\,|\,F\cap G)=1\neq\mathbb{P}(E)$.

Example: Indepdendent trials

A sequence of n independent trials results in a **success** with probability p and a failure with probability $1-p$. What is the probability that

- at least one success occurs?

- exactly k successes occur?

Gambler's Ruin Problem. A and B play an independent sequence of games. Each game, the winner gets one dollar from the loser, and play continues until one player is bankrupt. A starts with i dollars and B starts with $N - i$ dollars. A wins each game with probability p. What is the probability that A ends up with all the money?

Conditional probability obeys the axioms

Let $\mathbb{Q}_F(E) = \mathbb{P}(E \mid F)$. Then

- $0 \leq \mathbb{Q}_F(E) \leq 1$
- $\mathbb{Q}_F(\mathbb{S}) = 1$
- If $E_1, E_2, \ldots$ are disjoint, then

$$\mathbb{Q}_F(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{Q}_F(E_i)$$

Since $\mathbb{Q}_F$ obeys the axioms, all our previous propositions for probabilities give analogous results for conditional probabilities.

Examples

$\mathbb{P}(E^c \mid F) = 1 - \mathbb{P}(E \mid F)$

$\mathbb{P}(A \cup B \mid F) = \mathbb{P}(A \mid F) + \mathbb{P}(B \mid F) - \mathbb{P}(A \cap B \mid F)$

Example: Insurance policies re-visited

Insurance companies categorize people into two groups: accident prone (30%) or not. An accident prone person will have an accident within one year with probability 0.4; otherwise, 0.2. What is the conditional probability that a new policyholder will have an accident in his second year, given that the policyholder has had an accident in the first year?