5. Continuous Random Variables

Continuous random variables can take **any value in an interval**. They are used to model physical characteristics such as time, length, position, etc.

Examples

(i) Let X be the length of a randomly selected telephone call.

(ii) Let X be the volume of coke in a can marketed as 12oz.

Remarks

- A continuous variable has infinite precision, hence $\mathbb{P}(X = x) = 0$ for any x.
- In this case, the p.m.f. will provide no useful information.

Definition. X is a **continuous random variable** if there is a function $f(x)$ so that for any constants a and b, with $-\infty \leq a \leq b \leq \infty$,

$$\boxed{\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx} \tag{1}$$

- For δ small, $\mathbb{P}(a \leq X \leq a+\delta) \approx f(a)\,\delta$.
- The function $f(x)$ is called the **probability density function** (p.d.f.).
- For any a,
  $\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)\,dx = 0$.
- A discrete random variable does not have a density function, since if a is a possible value of a discrete RV X, we have $\mathbb{P}(X = a) > 0$.
- Random variables can be partly continuous and partly discrete.

The following properties follow from the axioms:

- $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- $f(x) \geq 0$.

Example. For some constant c, the random variable X has probability density function

$$f(x) = \begin{cases} cx^n & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) c and (b) $\mathbb{P}(X > x)$ for $0 < x < 1$.

Discussion problem. Let X be the duration of a telephone call in minutes and suppose X has p.d.f. $f(x) = c \cdot e^{-x/10}$ for $x \geq 0$. Find c, and also find the chance that the call lasts less than 5 minutes.

Cumulative Distribution Function (c.d.f.)

- The c.d.f. of a continuous RV is defined exactly the same as for discrete RVs:
  $$\boxed{F(x) = \mathbb{P}(X \leq x) = \mathbb{P}\left(X \in (-\infty, x]\right)}$$
- Hence $F(x) = \int_{-\infty}^{x} f(x)dx$, and differentiating both sides we get
  $$\boxed{\frac{dF}{dx} = f(x)}$$

Example. Suppose the lifetime, X, of a car battery has $\mathbb{P}(X > x) = 2^{-x}$. Find the p.d.f. of X.

Expectation of Continuous RVs

Definition. For X a continuous RV with p.d.f. $f(x)$

$$\boxed{\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx}$$

- Intuitively, this comes from the discrete case by replacing $\sum$ with $\int$ and $p(x_i)$ with $f(x)dx$.
- We can think of a continuous distribution as being approximated by a discrete distribution on a lattice $\dots, -2\delta, -\delta, 0, \delta, 2\delta, \dots$ for small δ.

Exercise. Let X be a continuous RV and let X_δ be a discrete RV approximating X on the lattice $\dots, -2\delta, -\delta, 0, \delta, 2\delta, \dots$ for small δ. Sketch a p.d.f. $f(x)$ for X and the corresponding p.m.f $p(x)$ for X_δ, paying attention to the scaling on the y-axis.

Example. Suppose X has p.d.f.

$$f(x) = \begin{cases} (1-x^2)(3/4) & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

Find the expected value of X.

Solution.

Note. Similarly to discrete RVs, the expected value is the balancing point of the graph of the p.d.f., and so if the p.d.f. is symmetric then the expected value is the point of symmetry. A sketch of the p.d.f. quickly determines the expected value in this case:

Example. The density function of X is

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$$

If $\mathbb{E}(X) = 3/5$, find a and b.

Solution.

Proposition 1. For X a non-negative continuous RV, with p.d.f. f and c.d.f. F,

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)dx = \int_0^\infty (1 - F(x))\, dx$$

Proof.

Proposition 2. For X a continuous RV with p.d.f. f and any real-valued function g

$$\mathbb{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Proof.

Example. A stick of length 1 is split at a point U that is uniformly distributed over $(0, 1)$. Determine the expected length of the piece that contains the point p, for $0 \leq p \leq 1$.

Solution.

Variance of Continuous RVs

- The definition is the same as for discrete RVs:

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right]$$

- The basic properties are also the same

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \left(\mathbb{E}(X)\right)^2 \\
\mathbb{E}(aX + b) &= a\mathbb{E}(X) + b \\
\mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)
\end{aligned}$$

Example. If $\mathbb{E}(X) = 1$ and $\mathrm{Var}(X) = 5$, find

(a) $\mathbb{E}\left[(2 + X)^2\right]$

(b) $\mathrm{Var}(4 + 3X)$

The Uniform Distribution

Definition: X has the **uniform distribution** on $[0, 1]$ (and we write $X \sim \text{Uniform}[0, 1]$ or just $X \sim \text{U}[0, 1]$) if X has p.d.f.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

- This is what people usually mean when they talk of a "random" number between 0 and 1.
- Each interval of length δ in $[0, 1]$ has equal probability: $\int_x^{x+\delta} f(y)dy = \delta$
- The chance of X falling into an interval is equal to the length of the interval.

Generalization: X has the **uniform distribution** on $[a, b]$ (i.e. $X \sim \mathrm{U}[a, b]$) if X has p.d.f.

$$f(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

Proposition. If $Z \sim \mathrm{U}[0, 1]$, then $(b-a)Z + a \sim \mathrm{U}[a, b]$.

Proof.

Example. Let $X \sim \mathrm{U}[a,b]$. Show that $\mathbb{E}(X) = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$,

(a) by standardizing (i.e., using the previous proposition);

(b) directly (by brute force).

Example. Your company must make a sealed bid for a construction project. If you succeed in winning the contract (by having the lowest bid), then you plan to pay another firm $\$100,000$ to do the work. If you believe that the maximum bid (in thousands of dollars) of the other participating companies can be modeled as being the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your expected profit?

Solution.

Discussion problem. Suppose $X \sim \mathrm{U}[5, 10]$. Find the probability that $X^2 - 5X - 6$ is greater than zero.

Solution.

The Standard Normal Distribution

Definition: Z has the **standard normal distribution** if it has p.d.f.

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$$

- $f(x)$ is symmetric about $x = 0$, so $\mathbb{E}(X) = 0$.
- $\mathrm{Var}(X) = 1$. Check this, using integration by parts:

The Normal Distribution

Definition. If X has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\}$$

then X has the **normal distribution**, and we write $X \sim \mathrm{N}(\mu, \sigma^2)$.

- $\mathbb{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.
- If $Z \sim \mathrm{N}(0,1)$ then Z is **standard normal**.

Proposition. Let $Z \sim \mathrm{N}(0,1)$ and set $X = \mu + \sigma Z$ for constants μ and σ. Then, $X \sim \mathrm{N}(\mu, \sigma^2)$.

Proof.

Exercise. Check that the standard normal density integrates to 1.

Trick. By changing to polar coordinates, show

$$\left(\int_{-\infty}^{\infty} \exp\{-x^2/2\}\, dx\right)^2 = 2\pi.$$

Solution.

Calculating with the Normal Distribution

- There is no closed form solution to the integral $\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, so we rely upon computers (or tables).

- The c.d.f. of the standard normal distribution is
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$
This is tabulated on page 201 of Ross.

Example. Find $\int_{-1}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Solution.

Standardization

If we wish to find $\mathbb{P}(a \leq X \leq b)$ where $X \sim \mathrm{N}(\mu, \sigma^2)$, we write $X = \mu + \sigma Z$. Then,

$$\begin{aligned}\mathbb{P}(a \leq X \leq b) &= \mathbb{P}\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\end{aligned}$$

Example. Suppose the weight of a new-born baby averages 8 lbs, with a SD of 1.5 lbs. If weights are normally distributed, what fraction of babies are between 7 and 10 pounds?

Solution.

Example continued. For what value of x does the interval $[8 - x, 8 + x]$ include 95% of birthweights?

Solution.

Example. Suppose that the travel time from your home to your office is normally distributed with mean 40 minutes and standard deviation 7 minutes. If you want to be 95% percent certain that you will not be late for an office appointment at 1 p.m., What is the latest time that you should leave home?

Solution.

The Normal Approximation to the Binomial Distribution

- If $X \sim \text{Binomial}(n, p)$ and n is large enough, then X is **approximately** $\text{N}(np, np(1-p))$.
- Rule of thumb: this approximation is reasonably good for $np(1-p) > 10$
- $\mathbb{P}(X = k) \approx \mathbb{P}(k - 1/2 < Y < k + 1/2)$ where $Y \sim \text{N}(np, np(1-p))$.
- Note: $\mathbb{P}(X \leq k)$ is usually approximated by $\mathbb{P}(Y < k + 1/2)$.

Example. I toss 1000 coins. Find the chance (approximately) that the number of heads is between 475 and 525, inclusive.

Solution.

Example. The ideal size of a first-year class at a particular college is 150 students. The college, knowing from past experience that on the average only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.

Solution.

The Exponential Distribution

Definition. X has the **exponential distribution** with parameter λ if it has density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- We write $X \sim \text{Exponential}(\lambda)$.
- $\mathbb{E}(X) = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.
- $F(x) = 1 - e^{-\lambda x}$

Example. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the probability that

(a) the computer will break down within the first 100 hours;

(b) given that it it still working after 100 hours, it breaks down within the next 100 hours.

Solution.

The Memoryless Property of Exponential RVs

- The exponential distribution is the continuous analogue of the geometric distribution (one has an exponentially decaying p.m.f., the other an exponentially decaying p.d.f.).
- Suppose that $X \sim \text{Exponential}(\lambda)$. Then $\mathbb{P}(X > t + s | X > t) = e^{-\lambda s} = \mathbb{P}(X > s)$.

  Check this:

- This is an analog for continuous random variables of the memoryless property that we saw for the geometric distribution.

Example. At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

Solution.

Example. Suppose $X \sim \mathrm{U}[0,1]$ and $Y = -\ln(X)$ (so $Y > 0$). Find the p.d.f. of Y.

Solution.

Note. This gives a convenient way to simulate exponential random variables.

- Calculations similar to the previous example are required whenever we want to find **the distribution of a function of a random variable**

Example. Suppose X has p.d.f. $f_X(x)$ and $Y = aX + b$ for constants a and b. Find the p.d.f. $f_Y(y)$ of Y.

Solution.

Example. A stick of length 1 is split at a point U that is uniformly distributed over $(0, 1)$. Determine the p.d.f. of the longer piece.

Solution.