2. Combinatorics: the systematic study of counting

The Basic Principle of Counting (BPC)

Suppose r experiments will be performed. The 1st has n_1 possible outcomes, for each of these outcomes there are n_2 possibilities for the 2nd, etc.

- The total # of outcomes for all r experiments combined is $\boxed{n_1 \times n_2 \times \cdots \times n_r}$
- The BPC tells us how to count leaves on a tree.

Example 1. Draw the tree for $r = 3$, $n_1 = 3$, $n_2 = n_3 = 2$.

Example 2. (k-tuples)

- An **ordered list** of k elements $(z_1, \ldots, z_k)$ is called a k-tuple.
- By BPC, if there are n_1 choices for z_1, n_2 choices for z_2, etc., then the number of possible k-tuples is $n_1 \times n_2 \times \cdots \times n_k$.

Example 3. If license plates have numbers in the first three places, followed by three letters, how many different plates are possible?

Example 4. (k-tuples without repetition)

- The BPC can also be used to count the number of such license plates if no letter or number can be repeated:

Example 5. (Sampling with replacement)

- A box contains n balls labeled $1, \ldots, n$. We draw a ball at random, note its number, and then replace it. Repeating k times gives a list $(i_1, \ldots, i_k)$.

- The sample space for this experiment is
$\mathbb{S} = \{\text{all } k\text{-tuples with entries } 1, \ldots, n\}$
which has $\#\mathbb{S} = n^k$.

- If we assume all n^k outcomes are equally likely, we say we have a **random sample** of size k **drawn with replacement** from a **population** of size n.

Example 6. Rolling a die k times gives a random sample with replacement for $n = 6$.

Example 7. A die is rolled four times. What is the probability of getting at least one $\boxed{6}$?

Solution. **Whenever you see "at least one" think about the complement "none"**

Note. What is wrong with reasoning as follows:

Let $E_i = \{\,\boxed{6} \text{ on roll } i\}$, and set $E = \bigcup_{i=1}^{4} E_i$.
Then, $\mathbb{P}(E) = \mathbb{P}(\bigcup_{i=1}^{4} E_i) = 4 \times \frac{1}{6} = 2/3$.

Sampling without replacement

- A box contains n balls labeled $1, \ldots, n$. If we draw k times, without replacing balls between draws, the outcome can still be written as a k-tuple $(i_1, \ldots, i_k)$ with i_j being the outcome of draw j.
- What is the sample space? Use the BPC to count this set.

- Recall that $n!$ (n factorial) is defined by $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$, when n is a positive integer. We also define $0! = 1$.

Example: 3 balls are drawn at random from an urn containing 8 red balls and 12 black balls. The draws are made without replacement. Find the chance that all 3 balls drawn are black.

Solution:

Permutations and Combinations

- A **permutation** is an **ordering** of a set of objects. Suppose the objects are labeled $1, 2, \ldots, n$, then an ordering is an n-tuple with no repeats. This is like sampling n times without replacement, so

  $$\boxed{\#\ \text{permutations} = n(n-1)\ldots 1 = n!}$$

- A **combination** is an **unordered** selection of objects. Write $\binom{n}{k}$ for the number of ways to choose k objects from n, which we call "n choose k." The BPC can be used to derive the formula

  $$\boxed{\binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$$

Proof:

- Counting problems may involve breaking down the enumeration into a sequence of easier problems. Formally, this involves the BPC.

Example. How many committees with 2 Republican, 2 Democrat and 3 Independent senators can be formed from a group of $5R$, $6D$ and $4I$?

Solution.

Example: Poker. A poker hand consists of 5 cards dealt from a shuffled deck of 52. So the number of possible hands is $\binom{52}{5} = 2,598,960$, and they are all equally likely. Find the chance of ...

(i) A pair (two cards of the same rank, all others of different ranks).

(ii) A straight (cards form a sequence and not all of the same suit).

Example: The birthday problem.

If 25 strangers are in a room, what is the chance that at least two of them share a birthday?

Example. An instructor gives her class 10 questions and promises to select 5 at random for hte final. What is the chance that a student who can solve 7 of them will be able to do the whole final?

Solution.

Discussion problem: Blackjack. Find the probability that two cards dealt from a shuffled deck form a blackjack (an A together with a 10, J, Q or K).

Solution. Explain a principled method, as well as looking for the right answer!

Permutations with indistinguishable objects

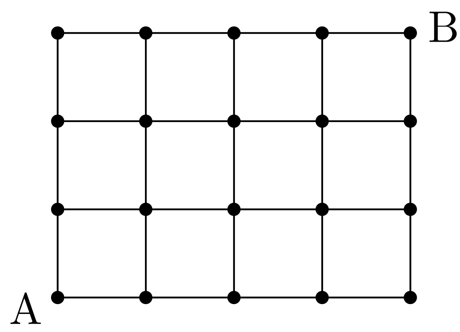
- Suppose we want to count the number of arrangements of the letters STATISTICS. We use a method similar to the proof of the formula for $\binom{n}{k}$, proceeding as follows:
- First, suppose the letters are distinguishable, by adding labels. Then the arrangements can be counted directly:
- Next, count the labeled arrangements a different way, by adding labels to the unlabeled arrangements:

Discussion Problem. Delegates from 10 countries are to be seated in a row. How many arrangements are possible if the American delegate must sit next to the Brazilian, and the Chinese delegate must not sit next to the Dutch?

Solution. Arrange blocks of objects and then label the blocks.

Hint. Consider counting complementary events ("must sit next to" versus "must not sit next to").

Example. How many possible paths are there from A to B on the grid below, if at each step you can go one step up or one step to the right?



Solution.

Ordered versus unordered selections

- Roll 5 dice. Let $\{1, 2, 3, 4, 5\}$ be the event that one die shows $\boxed{1}$, one of them shows $\boxed{2}$, etc. This event is written as an **unordered** set.

- If the dice are indistinguishable, we can only observe unordered outcomes.

- Now suppose the dice are colored red, white, blue, green, yellow. Let $(1, 2, 3, 4, 5)$ correspond to red showing $\boxed{1}$, white showing $\boxed{2}$, etc. This event is written as an **ordered** k-tuple.

- Compute the following:

$$\mathbb{P}\big[(1, 2, 3, 4, 5)\big] =$$

$$\mathbb{P}\big[\{1, 2, 3, 4, 5\}\big] =$$

$$\mathbb{P}\big[(5, 5, 5, 5, 5)\big] =$$

$$\mathbb{P}\big[\{5, 5, 5, 5, 5\}\big] =$$

Note. When sampling with replacement, not all unordered sets are equally likely!

Example. 8 castles (i.e., rooks) are randomly placed on a chess board. Find the chance that no rook can capture another (i.e., no two rooks are on the same rank or file).

Solution 1. Label the rooks $R_1, R_2, \ldots, R_8$ and the squares as $1, \ldots, 64$.

- Define $\mathbb{S}$, and count it.

- Let $E = \{\text{no two rooks can capture each other}\}$. Count E, by placing rooks in turn, to find $\mathbb{P}(E)$.

Solution 2. Keep the rooks indistinguishable.

- Define $\mathbb{S}$, and count it.

- Count the outcomes in E for this different sample space.

Note. Solution 2 is harder to carry out than Solution 1. Labeling indistinguisable objects often helps, but not always!

Discussion Problem. Ten students divide themselves randomly into two teams, to play five-a-side soccer. Find the chance that Xuan, Yasmin and Zack are all on the same team.

Solution.

Multinomial coefficients

- Suppose we want to divide n objects into r groups, labeled $1, 2, \ldots, r$, with n_i objects in group i for $i = 1, 2, \ldots r$ and $\sum_{i=1}^{n} r_i = n$. In how many ways can this be done?
- The number of such arrangements is called "n choose $n_1, \ldots, n_r$" and written as $\binom{n}{n_1\; n_2\; \ldots\; n_r}$.
- Counting this **multinomial coefficient** gives

$$\boxed{\binom{n}{n_1\; n_2\; \ldots\; n_r} = \frac{n!}{n_1!\, n_2! \ldots, n_r!}}$$

Proof. All $n!$ permutations of $1, \ldots, n$ can be counted by first assigning objects to groups and then assigning labels within each group, writing permutations as $(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{rn_r})$. Now apply the BPC:

Another way to count multinomial coefficients

- To divide n objects into r groups of size $n_1, \ldots, n_r$, we could note that there are $\binom{n}{n_1}$ ways to pick the first group, then $\binom{n-n_1}{n_2}$ ways to choose the second from the remaining $n - n_1$ objects, etc.
- Now apply the BPC:

Example. 12 students are divided into three groups of sizes 3, 4 and 5 at random. What is the chance that Ankur and Betty are in the same group?

Solution.