6. Jointly Distributed Random Variables

We are often interested in the relationship between two or more random variables. Example: A randomly chosen person may be a smoker and/or may get cancer.

Definition. X and Y are **jointly distributed random variables** if X and Y are both random variables defined on the same sample space $\mathbb{S}$.

Definition. X and Y are **discrete jointly distributed** RVs if they both have a countable set of possible values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$

- In this case, the **joint probability mass function** of X and Y is $p(x_i, y_j) = \mathbb{P}(\{X = x_i\} \cap \{Y = y_j\})$.
- X and Y each have their own p.m.f., which are usually written $p_X(x)$ and $p_Y(y)$ and are called the **marginal** p.m.f.s of X and Y.
- The joint p.m.f of X and Y may be written as $p_{XY}(x, y)$.

Example. Two cards are dealt from a shuffled deck. Let X be the number of these cards which are spades, and Y be the number of hearts. Find the joint p.m.f. of X and Y.

Solution.

Example. Two fair dice are rolled. X is the value on the first die and Y is the larger of the two values. Find the joint p.m.f. of X and Y.

Solution.

Properties of the p.m.f.

- The sum of the joint p.m.f. over the possible values of (X, Y) should be 1, i.e.,
  $\boxed{\sum_{ij} p_{XY}(x_i, y_j) = 1}$
- $\{X = x_i\} = \bigcup_j \{X = x_i, Y = y_j\}$. This lets us find the marginal p.m.f. from the joint p.m.f.,
  $\boxed{p(x_i) = \mathbb{P}(X = x_i) = \sum_j p(x_i, y_j)}$
- Writing the joint p.m.f of X and Y as a table, the marginal p.m.f. of X and Y can be found by summing the values in each row and column.

Continuous Joint Random Variables

Definition: X and Y are **continuous** jointly distributed RVs if they have a **joint density** $f(x, y)$ so that for any constants a_1, a_2, b_1, b_2,

$$\boxed{\mathbb{P}\big(a_1 < X < a_2, b_1 < Y < b_2\big) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\, dy\, dx}$$

- Almost any subset of $R \subset \mathbb{R}^2$ of practical interest can be approximated as the union of disjoint rectangles, and so

$$\mathbb{P}\big\{(X, Y) \in R\big\} = \int \int_R f(x, y)\, dy\, dx$$

- Integrating over the whole x-y plane,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

- The **marginal density** $f_X(x)$ of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

Example. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) $\mathbb{P}(X > 1, Y > 1)$; (b) $\mathbb{P}(X < Y)$.

Solution.

Example. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of the random variable $Z = X/Y$.

Solution.

Example. Suppose X and Y have joint p.d.f.

$$f(x,y) = \begin{cases} c(2x-y)^2 & 0 \le x, y \le 1 \\ 0 & \text{else} \end{cases}$$

Find c.

Solution.

Example. Suppose X and Y have joint density

$$f(x, y) = \begin{cases} 1/\pi & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{if } x^2 + y^2 > 1 \end{cases}$$

Find (a) $\mathbb{P}(X^2 + Y^2 < 1/2)$ and (b) the marginal density function of X .

Solution.

Joint Cumulative Distribution Functions

Definition. The **joint c.d.f.** of X and Y is

$$F(a,b) = \mathbb{P}(X \le a, Y \le b)$$

with **marginal c.d.f.s**

$$F_X(a) = \mathbb{P}(X \le a), \quad F_Y(b) = \mathbb{P}(Y \le b).$$

- If X and Y are jointly continuous, with density function $f(x,y)$ then

$$\mathbb{P}(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\, dy\, dx$$

- Differentiating both sides with respect to a and b, we obtain

$$\frac{\partial}{\partial a}\frac{\partial}{\partial b} F(a,b) = f(a,b).$$

Independence of RVs

Definition. X and Y are **independent** if, for any sets A and B, $\{X \in A\}$ is independent of $\{Y \in B\}$, i.e.

$$\boxed{\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)}$$

- Taking $A = (-\infty, a]$ and $B = (-\infty, b]$ gives $F(a, b) = F(a)F(b)$.
- For X and Y discrete, taking $A = \{x\}$, $B = \{y\}$ gives
  $p(x, y) = p_X(x)p_Y(y)$.
- For X and Y continuous, taking derivatives gives
  $f(x, y) = f_X(x)f_Y(y)$.

Example. Two people, A and B, agree to meet around 1pm. Each person arrives at a time uniformly distributed between $12:50$ and $1:10$, waits for 5 minutes and leaves if the other has not arrived. Find the chance they meet.

Solution.

Example. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. Find the probability that the distance between the two points is greater than $L/3$.

Solution.

Proposition

- Joint RVs X and Y are independent **if and only if** $F(x, y) = F_X(x)\, F_Y(y)$.
- Discrete RVs X and Y are independent **if and only if** $p(x, y) = p_X(x) p_Y(y)$.
- Continuous RVs X and Y are independent **if and only if** $f(x, y) = f_X(x) f_Y(y)$.
- Continuous/discrete RVs X and Y are independent **if and only if** their joint p.m.f./p.d.f. can be written as

$$\begin{aligned} p(x_i, y_j) &= h(x_i) g(y_j) \\ f(x, y) &= h(x) g(y) \end{aligned}$$

Proof. We've already seen the "only if." Check the "if" for "Continuous RVs X and Y are independent if and only if $f(x, y) = f_X(x) f_Y(y)$."

Example 1. Suppose X and Y have joint density $f(x, y) = c \cdot e^{-(x+y)}$ for $0 < x < 1$, $0 < y < 1$.

Are X and Y independent? Why?

Solution.

Example 2. Now let X and Y have joint density $f(x, y) = c \cdot e^{-(x+y)}$ for $0 < x < 1$, $0 < y < x$.

Are X and Y independent? Why?

Solution.

Sums of Independent RVs

The general case for continuous RVs: Let X and Y be independent, with densities f_X and f_Y, and c.d.f.s F_X and F_Y. Denote f_{X+Y} the p.d.f. of $X+Y$, and F_{X+Y} the c.d.f. of $X+Y$.

1. Compute the c.d.f. of the sum, $F_{X+Y}(z)$.

2. Differentiate to get the p.d.f., $f_{X+Y}(z)$.

Note: $\int_{-\infty}^{\infty} h(x-y)g(y)dy$ is called the **convolution** of $h(x)$ and $g(x)$.

Example. If X and Y are independent random variables, both uniformly distributed on $(0, 1)$, calculate the probability density of $X + Y$.

Solution.

Example. Suppose X and Y are independent, with $f_X(x) = \lambda e^{-\lambda x}, x > 0$ and $f_Y(y) = \lambda e^{-\lambda y}, y > 0$. Let $Z = X + Y$. Find the p.d.f. of Z.

Solution.

Example. Suppose X and Y are independent, with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Show that, for $Z = X + Y$, $Z \sim \text{Poisson}(\lambda + \mu)$.

Solution.

Example. The expected number of typographical errors on a page of a certain magazine is 0.2. What is the probability that an article of 10 pages contains (a) 0, and (b) 2 or more typographical errors?

Solution.

Sum of Normal RVs. Let X and Y be independent, with $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Y \sim \mathrm{N}(\nu, \tau^2)$. Then

$$\boxed{X + Y \sim \mathrm{N}(\mu + \nu, \sigma^2 + \tau^2)}$$

Solution. (a) First show the simpler case, that if $X \sim \mathrm{N}(0, \sigma^2)$ and $Y \sim \mathrm{N}(0, \tau^2)$ then $X + Y \sim \mathrm{N}(0, \sigma^2 + \tau^2)$.

Solution continued. (b) Extend (a) to solve the general case.

Example. The gross weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. What is the probability that the total gross sales over the next 2 weeks exceeds \$5000.

Solution.

Example. 25.2% of U.S. males and 23.6% of females never eat breakfast. Find the chance that in a random sample of 200 men and 200 women, at least as many women never east breakfast.

Solution.

Conditional Distributions

We want to be able to address questions like,

- "What is the distribution of the birth weight of a baby given that the mother weighs 120 lbs?"
- "What is the distribution of scores in the final given a score of 75% on the midterm?"

The Discrete Case. If X and Y are discrete, with joint probability mass function $p(x, y) = \mathbb{P}(X = x, Y = y)$, we define the **conditional p.m.f.** of X given $Y = y$ by

$$\begin{aligned} p_{X \mid Y}(x \mid y) &= \mathbb{P}(X = x \mid Y = y) \\ &= \frac{p(x, y)}{p_Y(y)} \end{aligned}$$

Example. Choose a number X at random from the set of numbers $\{1, 2, 3, 4, 5\}$. Now choose a number at random from the subset no larger than X, that is, from $\{1, \ldots, X\}$. Call this second number Y. Find the conditional mass function of X given that $Y = 3$.

Solution.

Example. If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2, calculate the conditional distribution of X given that $X + Y = n$.

Solution.

<u>The Continuous Case</u>.

- We must take care interpreting statements like $\mathbb{P}(X \in C \,|\, Y = y)$ when $\mathbb{P}(Y = y) = 0$.
- Instead, calculate $\mathbb{P}(X \in C \,|\, Y \in [y, y + \delta y])$ and let $\delta y \downarrow 0$.
- This leads to $\boxed{f_{X|Y}(x \,|\, y) = \dfrac{f(x,y)}{f_Y(y)}}$

<u>Derive this limit</u>:

Example. Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $\mathbb{P}(X > 1 \,|\, Y = y)$.

Solution.

Example. Let X_1, X_2 and X_3 be i.i.d. $\mathrm{U}[0,1]$.

(a) Find the chance that $X_3 > X_1 + X_2$.

(b) What is the chance that the largest of the three is greater than the sum of the other two?

Solution.

Bayes Theorem for Conditional RVs.

(a) For X and Y continuous RVs:

$$f_{X|Y}(x \,|\, y) = \frac{f_{Y|X}(y \,|\, x) f_X(x)}{f_Y(y)}$$

(b) The discrete case is similar, but with p.m.f.s instead of p.d.f.s:

$$p_{X|Y}(x \,|\, y) = \frac{p_{Y|X}(y \,|\, x) p_X(x)}{p_Y(y)}$$

Proof of (a):