8. Overview of further topics

The Weak Law of Large Numbers

- If $X_1, X_2, \ldots$ are i.i.d. RVs, with common mean $\mathbb{E}(X_i) = \mu$, then for any $\epsilon > 0$

$$\mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) \longrightarrow 0$$

  in the limit as $n \to \infty$. This is the **weak law of large numbers** (WLLN).

- The WLLN says that the sample average (for an i.i.d. sample of size n) converges to the expectation.

- This theorem can also be described as saying that the sample mean converges to the population mean.

- The WLLN says nothing about the rate of convergence. It also applies if $\mathrm{Var}(X) = \infty$.

Chebyshev's Inequality

- To prove the WLLN, we first obtain Chebyshev's inequality.

Proposition. If X is any RV, with $\mathbb{E}(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, then

$$\mathbb{P}\left(|X - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \quad \text{for any } \epsilon > 0$$

Proof.

Example. Suppose the daily change in value of stock is an i.i.d. sequence $X_1, X_2, \dots$ with $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$. What can you say, using Chebyshev's inequality, about the chance that the value changes by more than 5 in 10 days?

Solution.

Proof of Weak Law of Large Numbers

- $X_1, X_2, \ldots$ are i.i.d. with mean μ.
- Suppose an additional, unnecessary, condition that $X_1, X_2, \ldots$ have variance σ^2.
- Apply Chebyshev's inequality to $\bar{X}_n - \mu$ for

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

.

The Central Limit Theorem

- If $X_1, X_2, \ldots$ are i.i.d., with mean μ and variance σ^2, then for any constant a,

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right) \longrightarrow \mathbb{P}(Z \leq a)$$

in the limit as $n \to \infty$, where Z is standard normal, i.e. $Z \sim \mathrm{N}(0, 1)$.

Comments on the CLT

- The CLT may be re-written as

$$\frac{X_1 + \cdots + X_n - n\mu)}{\sigma\sqrt{n}} \longrightarrow Z$$

where the limit is interpreted as convergence of the c.d.f. (this type of limit is called convergence in distribution). This in turn can be rewritten as

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \longrightarrow Z.$$

- A remarkable thing about the CLT is that the behavior of the average depends only on the mean and the variance.

More Comments

- The CLT can be proved, but we can also view it as an empirical result. The CLT proposes a normal approximation for the distribution of an average; an approximation which can be tested by a computer experiment. How?

- The CLT "often" gives a good approximation for n as small as 10 or 20.

- The closer $X_1, X_2, \ldots$ are to having the normal distribution, the smaller the n required for a good approximation.

- If $X_1, X_2, \ldots$ are themselves i.i.d. $\mathrm{N}(\mu, \sigma^2)$ then it is exactly true that $\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu)$ has the standard normal distribution.

Example. A die is rolled 10 times. Use the CLT to approximate the chance that the sum is between 25 and 45.

Solution.

Review problems

Example. A sequence of independent trials is carried out, each with chance p of success. Let M be the number of failures preceeding the first success, and N the number of failures between the first two successes. Find the joint probability mass function of M and N.

Solution.

Example. A die is rolled repeatedly. Find the probability that the first roll is strictly greater than the next k rolls (i.e. if the values of the rolls are $X_1, X_2, \ldots$ then $X_1 > X_j$ for $j = 2, \ldots, k+1$).

Solution.

Example. A die is thrown N times. Let X be the number of times the die lands showing six spots, and Y the number of times it lands showing five spots. Find the mean and variance of $Z = X - Y$.

Solution.

Example. If X and Y are independent and identically distributed $\text{Uniform}[0,1]$ random variables, find the density of $Z = X/(X+Y)$.

Solution.

Example. Suppose X and Y have joint density

$$f(x,y) = ce^{-(x+y^2)}$$

on the region $x \geq 0$ and $-\infty < y < \infty$, with c being an unknown constant. Find the expected value of $X + Y^2$. You do not necessarily have to do any integration!

Solution.