Analysis of Time Series

Chapter 6: Extending the ARMA model: Seasonality, integration and trend

Edward L. Ionides

Outline

1. Seasonality
   - The SARMA model
2. Differencing and integration
   - The ARIMA model
   - The SARIMA model
3. Trend estimation: regression with ARMA errors

Seasonal autoregressive moving average (SARMA) models

- A general SARMA$(p,q) \times (P,Q)_{12}$ model for monthly data is

[S1] $$\phi(B)\Phi(B^{12})(Y_n - \mu) = \psi(B)\Psi(B^{12})\epsilon_n,$$

where $\{\epsilon_n\}$ is a white noise process and

$$\begin{aligned}
\mu &= \mathbb{E}[Y_n] \\
\phi(x) &= 1 - \phi_1 x - \dots - \phi_p x^p, \\
\psi(x) &= 1 + \psi_1 x + \dots + \psi_q x^q, \\
\Phi(x) &= 1 - \Phi_1 x - \dots - \Phi_P x^P, \\
\Psi(x) &= 1 + \Psi_1 x + \dots + \Psi_Q x^Q.
\end{aligned}$$

- SARMA is a special case of ARMA, where the AR and MA polynomials are factored into a **monthly** polynomial in B and an **annual polynomial** (also called **seasonal polynomial**) in B^{12}.
- Everything we learned about ARMA models (including assessing causality, invertibility and reducibility) also applies to SARMA.

Choosing the period for a SARMA model

- For the SARMA$(p, q) \times (P, Q)_{12}$ model, 12 is called the **period**.
- One could write a SARMA model for some period other than 12.
- A SARMA$(p, q) \times (P, Q)_4$ model could be appropriate for quarterly data.
- In principle, a SARMA$(p, q) \times (P, Q)_{52}$ model could be appropriate for weekly data, though in practice ARMA and SARMA may not work so well for higher frequency data.
- The seasonal period should be appropriate for the system being modeled. It is usually inappropriate to fit a SARMA$(p, q) \times (P, Q)_9$ model just because you notice a high sample autocorrelation at lag 9.

Consider the following two models:

[S2] $Y_n = 0.5 Y_{n-1} + 0.25 Y_{n-12} + \epsilon_n,$

[S3] $Y_n = 0.5 Y_{n-1} + 0.25 Y_{n-12} - 0.125 Y_{n-13} + \epsilon_n,$

Question 6.1. Which of [S2] and/or [S3] is a SARMA model?

For [S3], we can notice

$1 - \frac{1}{2}x - \frac{1}{4}x^{12} + \frac{1}{8}x^{13} = (1 - \frac{1}{2}x)(1 - \frac{1}{4}x^{12})$

So, [S3] can be written

$(1 - 0.5B)(1 - 0.25B^{12}) Y_n = \varepsilon_n$

[S2] lacks the B^{13} term, so cannot be written in a product form & is not SARMA.

Question 6.2. Why do we assume a multiplicative structure in the SARMA model, [S1]? What theoretical and practical advantages (or disadvantages) arise from requiring that an ARMA model for seasonal behavior has polynomials that can be factored as a product of a monthly polynomial and an annual polynomial?

The multiplicative assumption is convenient for data analysis. It separates the annual and local ("monthly") effects into two different parts of the ARMA polynomial – it separates the roots of the polynomial into local & annual components.

There is no clear scientific reason to prefer [S3] to the non-SARIMA [S2].

Perhaps there are computational advantages?

Fitting a SARMA model

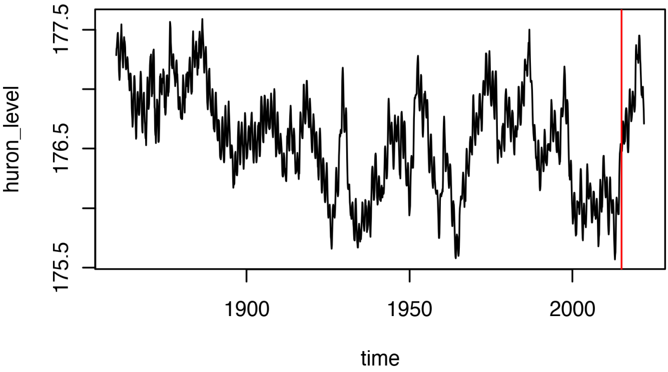
We fit a monthly version of the Lake Huron depth data described earlier.

```
dat <- read.table(file="huron_level.csv",sep=",",header=TRUE)
dat[1:3,1:7]
```

```
 year     jan     feb     mar     apr     may     jun
 1860 177.285 177.339 177.349 177.388 177.425 177.461
 1861 177.077 177.105 177.224 177.254 177.382 177.431
 1862 177.227 177.181 177.272 177.321 177.397 177.437
```

this collapses the matrix to a time series. This works because a matrix in R is a vector with a "dim" attribute, and with data stored column-wise.

```
huron_level <- as.vector(t(dat[,2:13]))
time <- rep(dat$year,each=12)+ rep(0:11,nrow(dat))/12
plot(huron_level~time,type="l")
```



Based on our previous analysis, we try fitting AR(1) for the annual polynomial. We try ARMA(1,1) for the monthly part, giving

$$(1 - \Phi_1 B^{12})(1 - \phi_1 B)Y_n = (1 + \psi_1 B)\epsilon_n. \quad (1)$$

- As discussed earlier, we analyze data only up to 2014, shown by a red line on the plot.

```
huron_level <- huron_level[time < 2014.99]
time <- time[time < 2014.99]
huron_sarma11x10 <- arima(huron_level,
  order=c(1,0,1),
  seasonal=list(order=c(1,0,0),period=12)
)
```

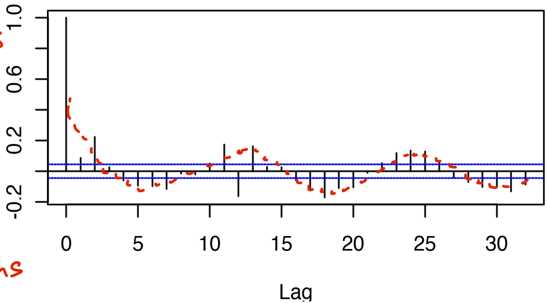
never use == to compare floating point numbers in R or any language.

```
huron_sarma11x10


Call:
arima(x = huron_level, order = c(1, 0, 1), seasonal = list(order = c(1, 0, 0),
    period = 12))

Coefficients:
         ar1     ma1    sar1  intercept
      0.9649  0.4168  0.5196   176.5727
s.e.  0.0062  0.0198  0.0215     0.0931

sigma^2 estimated as 0.002373:  log likelihood = 2977.19,  aic = -5944.37
```

Residual analysis

- Residual analysis is similar to non-seasonal ARMA models.
- We look for residual correlations at lags corresonding to multiples of the period (here, 12, 24, 36, ...) for misspecified annual dependence.

perhaps there are indications of an oscillating, decaying ACF. The period here is 12 months



Question 6.3. What do you conclude from this residual analysis? What would you do next? This suggests we need an AR(2) term to explain cyclical behavior of residuals. From our previous investigation of cycles in AR(2), this could be added to the local part.

ARMA models for differenced data

- Applying a difference operation to the data can make it look more stationary and therefore more appropriate for ARMA modeling.
- This can be viewed as a **transformation to stationarity**
- We can transform the data $y^*_{1:N}$ to $z_{2:N}$

$$z_n = \Delta y^*_n = y^*_n - y^*_{n-1}. \quad (2)$$

- Then, an ARMA(p,q) model $Z_{2:N}$ for the differenced data $z_{2:N}$ is called an **integrated autoregressive moving average** model for $y^*_{1:N}$ and is written as ARIMA(p,1,q).
- Formally, the ARIMA(p,d,q) model with intercept μ for $Y_{1:N}$ is

[S4] $\quad \phi(B)\big((1-B)^d Y_n - \mu\big) = \psi(B)\,\epsilon_n,$

where $\{\epsilon_n\}$ is a white noise process; $\phi(x)$ and $\psi(x)$ are ARMA polynomials.

- It is unusual to fit an ARIMA model with $d > 1$.

unit root: root with absolute value 1, i.e. on the complex unit circle.

- We see that an ARIMA(p,1,q) model is almost a special case of an ARMA(p+1,q) model with a **unit root** to the AR(p+1) polynomial.

Question 6.4. Why "almost" not "exactly" in the previous statement?

ARIMA: $\phi(B)[(1-B)^d Y_n - \mu] = \psi(B)\epsilon_n$ ①

ARMA with unit roots: $\phi(B)(1-B)^d[Y_n - \mu] = \psi(B)\epsilon_n$ ②

ARMA models are usually stationary & causal, so μ in ② is usually interpreted as a mean.

ARIMA models are not stationary. Consider the simplest case, ARIMA(0,1,0)

$(1-B)Y_n - \mu = \epsilon_n$ ③

$Y_n = Y_{n-1} + \mu + \epsilon_n$

③ is a random walk with drift: here, μ is the trend parameter, not a mean parameter.

Two reasons to fit an ARIMA(p,d,q) model with $d > 0$

1. You may really think that modeling the differences is a natural approach for your data. The S&P 500 stock market index analysis in Chapter 3 is an example of this, as long as you remember to first apply a logarithmic transform to the data.
2. Differencing often makes data look "more stationary" and perhaps it will then look stationary enough to justify applying the ARMA machinery.

- We should be cautious about this second reason. It can lead to poor model specifications and hence poor forecasts or other conclusions.
- The second reason was more compelling in the 1970s and 1980s. Limited computing power resulted in limited alternatives, so it was practical to force as many data analyses as possible into the ARMA framework and use method of moments estimators.

Practical advice on using ARIMA models

- ARIMA analysis is relatively simple to do. It has been a foundational component of time series analysis since the publication of the influential book "Time Series Analysis" (Box and Jenkins, 1970) which developed and popularized ARIMA modeling.
- A practical approach is:

  1. Do a competent ARIMA analysis.

  2. Identify potential limitations in this analysis and remedy them using more advanced methods.

  3. Assess whether you have in fact learned anything from (2) that goes beyond (1).

The SARIMA$(p,d,q)\times(P,D,Q)$ model

Combining integrated ARMA models with seasonality, we can write a general SARIMA$(p,d,q)\times(P,D,Q)_{12}$ model for nonstationary monthly data, given by

$$\text{[S5]} \quad \phi(B)\Phi(B^{12})\big[(1-B)^d(1-B^{12})^D Y_n - \mu\big] = \psi(B)\Psi(B^{12})\epsilon_n,$$

where $\{\epsilon_n\}$ is a white noise process, the intercept μ is the mean of the differenced process $\{(1-B)^d(1-B^{12})^D Y_n\}$, and we have ARMA polynomials $\phi(x)$, $\Phi(x)$, $\psi(x)$, $\Psi(x)$ as in model [S1].

- The SARIMA$(0,1,1)\times(0,1,1)_{12}$ model has often been used for forecasting monthly time series in economics and business. It is sometimes called the **airline model** after a data analysis by Box and Jenkins (1970).

Modeling trend with ARMA noise

- A general **signal plus noise** model is

[S6] $$Y_n = \mu_n + \eta_n,$$

  where $\{\eta_n\}$ is a stationary, mean zero stochastic process, and μ_n is the mean function.
- If, in addition, $\{\eta_n\}$ is uncorrelated, then we have a **signal plus white noise** model. The usual linear trend regression model fitted by least squares in Chapter 2 corresponds to a signal plus white noise model.
- We can say **signal plus colored noise** if we wish to emphasize that we're not assuming white noise.
- Here, **signal** and **trend** are used interchangeably. In other words, we are assuming a deterministic signal.
- At this point, it is natural for us to consider a signal plus ARMA(p,q) noise model, where $\{\eta_n\}$ is a stationary, causal, invertible ARMA(p,q) process with mean zero.
- As well as the $p + q + 1$ parameters in the ARMA(p,q) model, there will usually be unknown parameters in the mean function.

Linear regression with ARMA errors

- When the mean function (also known as the trend) has a linear specification,

$$\mu_n = \sum_{k=1}^{K} Z_{n,k}\beta_k, \tag{3}$$

  the signal plus ARMA noise model is known as **linear regression with ARMA errors**.
- Writing Y for a column vector of $Y_{1:N}$, μ for a column vector of $\mu_{1:N}$, η for a column vector of $\eta_{1:N}$, and Z for the $N \times K$ matrix with (n,k) entry $Z_{n,k}$, we have a general linear regression model with correlated ARMA errors,

$$Y = Z\beta + \eta. \tag{4}$$

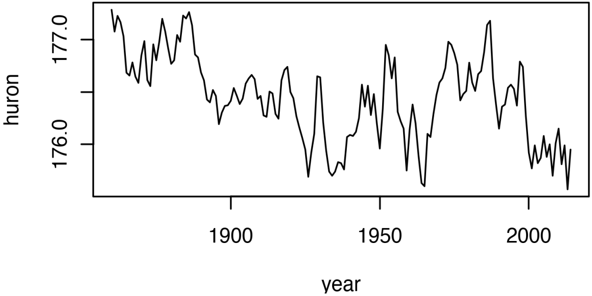
- From (4), $Y - Z\beta$ is ARMA so likelihood evaluation and numerical maximization can build on ARMA methods.

Inference for the linear regression model with ARMA errors

- Maximum likelihood estimation of $\theta = (\phi_{1:p}, \psi_{1:q}, \sigma^2, \beta)$ is a nonlinear optimization problem.
- Fortunately, `arima` in R can do it for us.
- As usual, we should look out for signs of numerical problems.
- Data analysis for a linear regression with ARMA errors model, using the framework of likelihood-based inference, is procedurally similar to fitting an ARMA model.
- This is a powerful technique, since the covariate matrix Z can include other time series. We can evaluate associations between different time series.
- With appropriate care (since **association is not causation**) we can draw inferences about mechanistic relationships between dynamic processes.

Evidence for systematic trend in Lake Huron level?

We return to annual data, say the January level, to avoid seasonality.



- Visually, there seems some evidence for a decreasing trend, but there are also considerable fluctuations.
- Let's test for a trend, using a regression model with Gaussian AR(1) errors. We have previously found that this is a reasonable model for these data.
- First, for comparison, we fit a null model with no trend.

```r
fit0 <- arima(huron,order=c(1,0,0))
```

```

Call:
arima(x = huron, order = c(1, 0, 0))

Coefficients:
         ar1  intercept
      0.8689   176.4577
s.e.  0.0408     0.1233

sigma^2 estimated as 0.04389:  log likelihood = 21.62,  aic = -37.25
```

- We compare `fit0` with a linear trend model, coded as `fit1`.
- The covariate is included via the `xreg` argument.

```
fit1 <- arima(huron,order=c(1,0,0),xreg=year)
```

```
Call:
arima(x = huron, order = c(1, 0, 0), xreg = year)

Coefficients:
         ar1  intercept     year
      0.8211   186.1652  -0.0050
s.e.  0.0455     3.6923   0.0019

sigma^2 estimated as 0.04244:  log likelihood = 24.37,  aic = -40.74
```

Setting up a formal hypothesis test

in regression language: μ is the "intercept"

arima() with xreg includes an intercept by default.

- To talk formally about these results, we must down a model and some hypotheses.
- Writing the data as $y^*_{1:N}$, collected at years $t_{1:N}$, the model we have fitted is

$$(1-\phi_1 B)(Y_n - \mu - \beta t_n) = \epsilon_n, \quad (5)$$

where $\{\epsilon_n\}$ is Gaussian white noise with variance σ^2. Our null model is

$$H^{\langle 0 \rangle} : \beta = 0, \quad (6)$$

and our alternative hypothesis is

$$H^{\langle 1 \rangle} : \beta \neq 0. \quad (7)$$

Question 6.5. How do we test $H^{\langle 0 \rangle}$ against $H^{\langle 1 \rangle}$?

- Construct two different tests using the R output above.
- Which test do you think is more accurate, and why?

1. Z-test on the "year" coefficient, the slope β in our model.
Test statistic: $\frac{\hat{\beta}}{SE(\hat{\beta})} = \frac{-0.0050}{0.0019} = -2.63$, pvalue: 2 pnorm(-2.63) = 0.0085
Significant at 1% level, rejecting the null hypothesis $\beta = 0$.

2. Likelihood ratio test (LRT)
Δ = difference in log likelihood = 24.37 - 21.62 = 2.75
χ^2 approximation, on 1 d.f. : $\frac{1}{2}\Delta \sim \chi^2_1$
p-value = 1 - pchisq(2*2.75, 1) = 0.019
This rejects the null hypothesis at 5% level but not 1%.
We found earlier that profile methods (equivalent to LRT) were more reliable than std errors from observed Fisher information.
We could check if the same holds here.

Question 6.6. How would you check whether your preferred test is indeed better? What other supplementary analysis could you do to strengthen your conclusions?

1. Check assumptions. Diagnostic plots. Both tests are based on the same model assumptions, but they might have different sensitivity to those assumptions
2. Bootstrap. Check the coverage of a bootstrap confidence interval. A parametric bootstrap simulating from the model would test asymptotic approximations but not model misspecification. We could also simulate with longer tails, to test non-normality.

Further reading

- Section 3.9 of Shumway and Stoffer (2017) discusses SARIMA modeling.
- Section 3.8 of Shumway and Stoffer (2017) introduces regression with ARMA errors.

References and Acknowledgements

Box GEP, Jenkins GM (1970). *Time Series Analysis: Forecasting and Control*. First edition. Holden–Day, San Francisco.

Shumway RH, Stoffer DS (2017). *Time Series Analysis and its Applications: With R Examples*. Springer.

- Compiled on January 31, 2022 using R version 4.1.2.

- We acknowledge previous versions of this course.