STATS 531 Homework 2

Due Tuesday 1/27, 11:59pm

Submit two files for your homework report via Canvas: (i) a pdf file; (ii) a Quarto (qmd) file that reproduces this pdf. Projects and subsequent homeworks will be done using qmd, so it is good to get started. The qmd source code for this assignment may be helpful. Quarto combines Python with Latex, so extra work will be required initially if you are unfamiliar with either of these. Quarto also lets you use R, but this semester STATS 531 is focusing on Python. You are reminded that the grading scheme for the homework report puts an emphasis on careful explanation of sources, as explained in the grading rubric.

Question 2.1. We investigate two ways to calculate the autocovariance function for AR and MA models. These ways are different from the two ways already demonstrated in the notes, but there is some overlap. The instructions below help you work through the case of a causal AR(1) model,

$$X_n = \phi X_{n-1} + \epsilon_n.$$

where $\{\epsilon_n\}$ is white noise with variance σ^2, and $-1 < \phi < 1$. Assume the process is stationary, i.e., it is initialized with a random draw from its stationary distribution. Show your working for both the approaches A and B explained below. If you want an additional challenge, you can work through the AR(2) or ARMA(1,1) case instead.

A. Using the stochastic difference equation to obtain a difference equation for the autocovariance function (ACF). Start by writing the ACF as

$$\gamma_h = \text{Cov}(X_n, X_{n+h}) = \text{Cov}(X_n, \phi X_{n+h-1} + \epsilon_{n+h}), \text{ for } h > 0.$$

Writing the right hand side in terms of γ_{h-1} leads to an equation which is formally a first order linear homogeneous recurrence relation with constant coefficients. To solve such an equation, we look for solutions of the form

$$\gamma_h = A\lambda^h.$$

Substituting this general solution into the recurrence relation, together with an initial condition derived from explicitly computing γ_0, provides an approach to finding two equations that can be solved for the two unknowns, A and λ.

B. Via a Taylor series calculation of the MA(∞) representation. Construct a Taylor series expansion of $g(x) = (1 - \phi x)^{-1}$ of the form

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + ...$$

Do this either by hand or using your favorite math software (if you use software, please say what software you used and what you entered to get the output). Use this Taylor series to write down the MA(∞) representation of an AR(1) model. Then, apply the general formula for the autocovariance function of an MA(∞) process.

C. Check your work for the specific case of an AR(1) model with $\phi_1 = 0.8$ by comparing your formula with the result of the Python function `statsmodels.tsa.arima_process.arma_acf` or `statsmodels.tsa.arima_process.arma.acovf`.

Question 2.2 Compute the autocovariance function (ACF) of the random walk model. Specifically, find the ACF, $\gamma_{mn} = \text{Cov}(X_m, X_n)$, for the random walk model specified by

$$X_n = X_{n-1} + \epsilon_n,$$

where $\{\epsilon_n\}$ is white noise with variance σ^2, and we use the initial value $X_0 = 0$.

Reading. The course notes are intended to be self-contained, and additional reading is therefore optional. We have covered much of the material through to Section 3.3 of Shumway and Stoffer "Time Series Analysis and its Applications" (pdf available via the UM library) or through Chapter 3 of Huang and Petukhina "Applied Time Series Analysis and Forecasting with Python" (pdf available via the UM library).