Statistics 620
Final exam, Fall 2011

Name: ____________________________ **UMID #:** ____________________________

- **There are 5 questions, each worth 10 points.**
- **You are allowed two single-sided sheets of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. Let $\{N(t)\}$ be a rate λ Poisson process, with arrival times $\{S_n, n = 0, 1, \ldots\}$. Evaluate the expectected sum of squares of the arrival times occurring before t,

$$E(t) = \mathbb{E}\Big[\sum_{n=1}^{N(t)} S_n^2\Big],$$

where we define $\sum_{n=1}^{0} S_n = 0$.

2. Consider two machines, operating simultaneously and independently, where both machines have an exponentially distributed fime to failure with mean $1/\mu$ (μ is the failure rate). There is a single repair facility, and the repair times are exponentially distributed with rate λ. What is the long run probability that no machine is operating?

3. Let $\{U_1, U_2, \dots\}$ be a sequence of independent, identically distributed Uniform random variables taking values in the interval $(0,1)$. Define $X_n = 2^n \prod_{k=1}^{n} U_k$ for $n > 1$, with $X_0 = 1$. Show that $\{X_n\}$ is a martingale, and discuss the limiting behavior of X_n and $\mathbb{E}[X_n]$ as n increases.

4. Let $\{B(t)\}$ be a standard Brownian motion, and define $Y(t) = e^{B(t)}/\big(1+e^{B(t)}\big)$. (a) Explain why $\{Y(t)\}$ is a diffusion process; (b) obtain its infinitesimal mean and variance; (c) explain whether or not sample paths of $\{Y(t)\}$ converge to a limit with probability one.

5. Let $\{B(t)\}$ be standard Brownian motion. The event that $\{B(t)\}$ has a zero crossing between s and t is $A(s,t) = \{B(u) = 0 \text{ for some } u \text{ with } s < u < t\}$. We wish to find $\mathbb{P}\{A(s,t)\}$. As a preliminary step, in part (a), we consider the hitting time $\tau_x = \min\{u \geq 0 : B(u) = x\}$ for $x > 0$. The calculation continues in part (b), on the following page.

(a) Find $\mathbb{P}\{\tau_x \leq t\}$ in terms of the standard normal distribution function, $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\{\frac{-u^2}{2}\}\, du$.

(b) By conditioning on $B(s)$, find an expression for $\mathbb{P}\{A(s,t)\}$. Evaluate this expression using the identity

$$\int_0^\infty e^{-v^2/2s} \left\{ \int_v^\infty e^{-u^2/2(t-s)}\, du \right\} dv = \sqrt{s(t-s)}\ \arccos \sqrt{s/t}, \tag{1}$$

where arccos is the inverse of the cosine function. You are not asked to prove (1) and so no specific knowledge about the arccos function is required for this question.