Statistics 620
Final exam, Fall 2012

Name: ______________________________ **UMID #:** ______________________________

- **There are 5 questions, each worth 10 points.**
- **You are allowed two single-sided sheets of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. This question asks you to confirm the validity of a variation on the Hastings-Metropolis algorithm. Let $Q = \{q_{ij}\}$ be a symmetric irreducible transition probability matrix (i.e., Q specifies the one-step transition probabilities of an irreducible, discrete time Markov chain with states $1, 2, \ldots$, and $q_{ij} = q_{ji}$). Let $a_1, a_2, \ldots$ be a sequence of positive numbers with $\sum_i a_i < \infty$. Define the Markov chain $\{X_n\}$ by the following transition rule:

Conditional on X_n, draw a random variable Y_{n+1} with $\mathbb{P}(Y_{n+1} = j | X_n = i) = q_{ij}$. Then, conditional on $X_n = i$ and $Y_{n+1} = j$, set

$$X_{n+1} = \begin{cases} j & \text{with probability } a_j/(a_i + a_j) \\ i & \text{with probability } a_i/(a_i + a_j) \end{cases}$$

Show that $\{X_n\}$ has limiting probabilities given by $\pi_j = a_j / \sum_i a_i$.

2. Taxis and customers arrive at a taxi station in accordance with independent Poisson processes, with respective rates one and two per minute. A taxi will wait, regardless of how many taxis are already in line, however a customer who does not find a taxi waiting leaves. Find

(a) the average number of taxis waiting.

(b) the proportion of arriving customers who get taxis.

3. Consider successive flips of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of flips until the sequence HTHTHT appears.

Note: this question is asking you explicitly to find a solution based on a martingale method.

Hint: recall that the martingale approach to this problem involves considering a sequence of gamblers who arrive at times $1, 2, \ldots$. Each gambler arrives with one dollar and bets it on the next flip landing H. Suppose that the gambler is offered fair odds for his bet. If he wins, he rolls all his capital into a subsequent bet on T, followed by H again and so forth. He quits the first time he loses or when HTHTHT appears. Let X_n be the combined profit (or loss) of all the gamblers up to, and including, the nth flip.

4. Let $B(t)$ be standard Brownian motion. For $0 < t_1 < t_2$, find an expression for $\mathbb{P}\{\max_{t_1 \le s \le t_2} B(s) > x\}$ in terms of the standard nornal distribution function, $\Phi(x) = \int_{-\infty}^{x} (1/\sqrt{2\pi}) e^{-u^2/2}\, du$, and the corresponding density function $\phi(x) = \frac{d}{dx}\Phi(x)$.
Hint: one approach is to use symmetry to find the distribution of $\max_{t_1 \le s \le t_2}(B(s) - B(t_1))$ and then to condition on $B(t_1)$.

5. Let $\{X(t)\}$ be an Ornstein-Uhlenbeck process with infinitesimal parameters $\mu(x,t) = -\alpha x$ and $\sigma^2(x,t) = 1$. Set $Y(t) = e^{X(t)}$.

(i) Argue that $Y(t)$ is a diffusion process (i.e., check that it satisfies an appropriate definition).

(ii) Find the infinitesimal mean and variance of $Y(t)$.