

Statistics 620
Final exam, Fall 2012

1. This question asks you to confirm the validity of a variation on the Hastings-Metropolis algorithm. Let $Q = \{q_{ij}\}$ be a symmetric irreducible transition probability matrix (i.e., Q specifies the one-step transition probabilities of an irreducible, discrete time Markov chain with states $1, 2, \dots$, and $q_{ij} = q_{ji}$). Let a_1, a_2, \dots be a sequence of positive numbers with $\sum_i a_i < \infty$. Define the Markov chain $\{X_n\}$ by the following transition rule:

Conditional on X_n , draw a random variable Y_{n+1} with $\mathbb{P}(Y_{n+1} = j | X_n = i) = q_{ij}$. Then, conditional on $X_n = i$ and $Y_{n+1} = j$, set

$$X_{n+1} = \begin{cases} j & \text{with probability } a_j / (a_i + a_j) \\ i & \text{with probability } a_i / (a_i + a_j) \end{cases}$$

Show that $\{X_n\}$ has limiting probabilities given by $\pi_j = a_j / \sum_i a_i$.

Solution:

$$\begin{aligned} P_{ij} &= \mathbb{P}(X_{n+1} = j | X_n = i) \\ &= \mathbb{P}(X_{n+1} = j | X_n = i, Y_{n+1} = j) \mathbb{P}(Y_{n+1} = j | X_n = i) \\ &= q_{ij} \frac{a_j}{a_i + a_j} \end{aligned}$$

One can then readily check that the detailed balance equations, $\pi_i P_{ij} = \pi_j P_{ji}$, are satisfied by $\pi_j = a_j / \sum_i a_i$, using the symmetric relationship that $q_{ij} = q_{ji}$. Also, it is immediate that $\sum_i \pi_i = 1$. Since the Markov chain is irreducible and aperiodic, we obtain that it is reversible and that π is the unique stationary distribution and the vector of limiting probabilities.

2. Taxis and customers arrive at a taxi station in accordance with independent Poisson processes, with respective rates one and two per minute. A taxi will wait, regardless of how many taxis are already in line, however a customer who does not find a taxi waiting leaves. Find

(a) the average number of taxis waiting.

(b) the proportion of arriving customers who get taxis.

Solution: (a) This is a birth-death process for the taxis, with birth rate $\lambda = 1$ and death rate $\mu = 2$. Now, using the balance equation,

$$\begin{aligned} P_n \lambda &= P_{n+1} \mu \\ \sum_{n=0}^{\infty} P_n &= 1 \end{aligned}$$

After some algebra,

$$P_k = \frac{1}{2^{k+1}}$$

Therefore, expected no. of taxis waiting = $\sum_{k=0}^{\infty} k P_k = 1$

(b) by the PASTA property,

$$\text{Proportion of customers who get taxi} = \mathbb{P}(\text{no. of taxis waiting is more than 1}) \quad (1)$$

$$= 1 - P_0 = 1/2 \quad (2)$$

3. Consider successive flips of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of flips until the sequence HTHTHT appears.

Note: this question is asking you explicitly to find a solution based on a martingale method.

Hint: recall that the martingale approach to this problem involves considering a sequence of gamblers who arrive at times $1, 2, \dots$. Each gambler arrives with one dollar and bets it on the next flip landing H. Suppose that the gambler is offered fair odds for his bet. If he wins, he rolls all his capital into a subsequent bet on T, followed by H again and so forth. He quits the first time he loses or when HTHTHT appears. Let X_n be the combined profit (or loss) of all the gamblers up to, and including, the n th flip.

Solution: Use a martingale approach similar to question 6.10 in Ross. The answer is $\mathbb{E}[N] = p^{-3}q^{-3} + p^{-2}q^{-2} + p^{-1}q^{-1}$ where $q = 1 - p$.

4. Let $B(t)$ be standard Brownian motion. For $0 < t_1 < t_2$, find an expression for $\mathbb{P}\{\max_{t_1 \leq s \leq t_2} B(s) > x\}$ in terms of the standard normal distribution function, $\Phi(x) = \int_{-\infty}^x (1/\sqrt{2\pi})e^{-u^2/2} du$, and the corresponding density function $\phi(x) = \frac{d}{dx}\Phi(x)$.

Hint: one approach is to use symmetry to find the distribution of $\max_{t_1 \leq s \leq t_2} (B(s) - B(t_1))$ and then to condition on $B(t_1)$.

Solution: let T be the time at which the maximum is attained, so $\max_{t_1 \leq s \leq t_2} B(s) = B_T$. Now,

$$\begin{aligned} \mathbb{P}(B_T \geq x | B(t_1) = a) &= \mathbb{P}(\max_{t_1 \leq s \leq t_2} (B(s) - B(t_1)) \geq x - a | B(t_1) = a) \\ &= \mathbb{P}(\max_{0 \leq s \leq t_2 - t_1} (B(s) \geq x - a)) \\ &= \mathbb{P}(T_{x-a} \leq t_2 - t_1) \\ &= 2\mathbb{P}(B(t_2 - t_1) \geq x - a) \\ &= 2(1 - \Phi(\frac{x - a}{\sqrt{t_2 - t_1}})) \\ &= 2\Phi(\frac{a - x}{\sqrt{t_2 - t_1}}) \end{aligned}$$

We note that,

$$\mathbb{P}(B_T \geq x | B(t_1) = a) = 1 \text{ if } a \geq x$$

Thus,

$$\begin{aligned} \mathbb{P}(B_T \geq x) &= \mathbb{E}[\mathbb{P}(B_T \geq x | B(t_1))] \\ &= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^x 2(1 - \Phi(\frac{x - a}{\sqrt{t_2 - t_1}}))e^{-a^2/2t_1} da + \mathbb{P}(B(t_1) > x) \\ &= \int_{-\infty}^x 2(1 - \Phi(\frac{x - a}{\sqrt{t_2 - t_1}}))\phi(\frac{a}{\sqrt{t_1}}) da + \Phi(-\frac{x}{\sqrt{t_1}}) \end{aligned}$$

5. Let $\{X(t)\}$ be an Ornstein-Uhlenbeck process with infinitesimal parameters $\mu(x, t) = -\alpha x$ and $\sigma^2(x, t) = 1$. Set $Y(t) = e^{X(t)}$.

(i) Argue that $Y(t)$ is a diffusion process (i.e., check that it satisfies an appropriate definition).

(ii) Find the infinitesimal mean and variance of $Y(t)$.

Solution:

(a). A diffusion is a continuous time, continuous state process with continuous sample paths that possesses the Markov property. An O-U process is a diffusion and $f(x) = e^x$ is a continuous function. Transforming the sample paths by a continuous function preserves the continuity of the sample paths. The transformation also does not affect the Markov property, so the transformed process is also a diffusion.

(b). We have

$$X(t) = \log Y(t)$$

Now, using the transformation formula with $f(x) = e^x$,

$$\begin{aligned}\mu_Y(y, t) &= \mu_X(x, t)f'(x) + \frac{1}{2}\sigma_X^2(x, t)f''(x) \\ &= -\alpha x e^x + \frac{1}{2}e^x \\ &= -\alpha y \log y + \frac{1}{2}y\end{aligned}$$

and

$$\begin{aligned}\sigma_Y^2(y, t) &= \sigma_X^2(x, t)[f'(x)]^2 \\ &= [e^x]^2 \\ &= y^2\end{aligned}$$