**Statistics 620**
**Final exam, Fall 2013**

**Name:** ________________________________ **UMID #:** ________________________________

- **There are 5 questions, each worth 10 points.**
- **You are allowed two single-sided sheets of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

| Problem | Points | Your Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total | 50 | |

**1**. Suppose that traffic on a road follows a Poisson process with rate $\lambda$ cars per minute. A chicken needs a gap of length at least $c$ minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time $T_{J-1}$ and complete his journey at time $T_{J-1} + c$.

(a) [4 points]. Suppose $T$ is exponentially distributed with rate $\lambda$. Find $\mathbb{E}[T \mid T < c]$.

Hint: Using the identity $\mathbb{E}[T] = \mathbb{P}(T < c)\,\mathbb{E}[T \mid T < c] + \mathbb{P}(T > c)\,\mathbb{E}[T \mid T > c]$ leads to a nice solution, though you can also solve the problem by direct calculation.

(b) [6 points] Use part (a) to show $\mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$. If you have not solved (a), you may leave your answer in terms of $\mathbb{E}[T \mid T < c]$.

**2**. We investigate a martingale solution to the same situation problem from Question 1. As before, traffic on a road follows a Poisson process with rate $\lambda$ cars per minute. A chicken needs a gap of length at least $c$ minutes in the traffic to cross the road. $t_1, t_2, t_3, \ldots$ are the interarrival times for the cars and $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time $T_{J-1}$ and complete his journey at time $T_{J-1} + c$. Note that $T_n - (n/\lambda)$ is a martingale.

(a) [3 points] Argue that $J$ is a stopping time for $t_1, t_2, \ldots$, and explain why $J - 1$ is not a stopping time.

(b) [7 points] Use a martingale argument to show that $\mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

**3**. We study a queue with impatient customers. Customers arrive at a single server as a Poisson process with rate $\lambda$ and require an exponential amount of service with rate $\mu$. customers waiting in line are impatient and if they are not in service they will leave at rate $\delta$ independent of their position in the queue. Show that for any $\delta > 0$ the system has a stationary distribution, and find an expression for this distribution.

**4**. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate $\lambda$. The customer and the dealer then disappear from the street for an amount of time with distribution $G$ while the transaction is completed. Customers that arrive during this time go away never to return.

(a) [5 points] At what rate does the dealer make sales? Explain your reasoning.

(b) [5 points] What fraction of customers are lost? Explain your reasoning.

**5**. Let $\{Z(t), 0 \leq t \leq 1\}$ be a Brownian bridge, i.e., a Gaussian diffusion with $\mathbb{E}[Z(t)] = 0$ and $\mathrm{Cov}(Z(s), Z(t)) = s \wedge t - st$. Define $X(t) = (1+t)Z(t/(1+t))$. Show that $\{X(t), t \geq 0\}$ is a standard Brownian motion.