Statistics 620
Final exam, Fall 2013

1. Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$.

(a) [4 points]. Suppose T is exponentially distributed with rate λ. Find $\mathbb{E}[T \,|\, T < c]$.

Hint: Using the identity $\mathbb{E}[T] = \mathbb{P}(T < c)\,\mathbb{E}[T \,|\, T < c] + \mathbb{P}(T > c)\,\mathbb{E}[T \,|\, T > c]$ leads to a nice solution, though you can also solve the problem by direct calculation.

Solution:

$$\begin{aligned}
\mathbb{E}(T|T<c) &= \frac{\mathbb{E}(T \,|\, T<c)}{\mathbb{P}(T<c)} \\
&= \frac{\int_0^c t\lambda e^{-\lambda t}dt}{1-e^{-c}} = \frac{-ce^{-\lambda c} + \frac{1}{\lambda}(1-e^{-\lambda c})}{1-e^{-c}} .
\end{aligned}$$

(b) [6 points] Use part (a) to show $\mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$. If you have not solved (a), you may leave your answer in terms of $\mathbb{E}[T \,|\, T < c]$.

Solution: Note that

$$T_{J-1} + c = c\ \mathbb{I}_{\{t_1 > c\}} + (t_1 + c + \sum_{k=2}^{J-1} t_k)\mathbb{I}_{\{t_1 \le c\}} .$$

Taking expectation on both sides above, we have

$$\mathbb{E}(T_{J-1}+c) = c\mathbb{P}(t_1 > c) + \mathbb{P}(t_1 \le c)\mathbb{E}\left(t_1 | t_1 \le c\right) + \mathbb{P}(t_1 \le c)\mathbb{E}\left((\sum_{k=2}^{J-1} t_k + c) \,\middle|\, t_1 \le c \right) . \quad (1)$$

Note that in above, by result in (a), the second term on the r.h.s. of (1) equals $-ce^{-\lambda c} + \frac{1}{\lambda}(1-e^{-\lambda c})$, and the third term equals $\mathbb{P}(t_1 \le c)\mathbb{E}(T_{J-1}+c)$ by memoryless property. Hence, it follows that

$$\begin{aligned}
\mathbb{E}(T_{J-1}+c) &= \frac{ce^{-\lambda c} + \left(-ce^{-\lambda c} + \frac{1}{\lambda}(1-e^{-\lambda c})\right)}{\mathbb{P}(t_1 > c)} \\
&= \frac{\frac{1}{\lambda}\left(1-e^{-\lambda c}\right)}{e^{-\lambda c}} = \frac{1}{\lambda}\left(e^{\lambda c} - 1\right) .
\end{aligned}$$

2. We investigate a martingale solution to the same situation problem from Question 1. As before, traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. $t_1, t_2, t_3, \ldots$ are the interarrival times for the cars and $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Note that $T_n - (n/\lambda)$ is a martingale.

(a) [3 points] Argue that J is a stopping time for $t_1, t_2, \ldots$, and explain why $J - 1$ is not a stopping time.

Solution: Let $X_n = T_n - \frac{n}{\lambda}$. J is a stopping time because

$$\begin{aligned}\{J = n\} &= \{T_1 < c, T_2 - T_1 < c, \ldots, T_{n-1} - T_{n-2} < c, T_n - T_{n-1} > c\} \\ &= \{X_1 + \frac{1}{\lambda} < c, X_2 - X_1 + \frac{1}{\lambda} < c \ldots, X_{n-1} - X_{n-2} + \frac{1}{\lambda} < cX_n - X_{n-1} > c\},\end{aligned}$$

which is determined by the value of $X_1, \ldots, X_n$. Similarly, $\{J-1 = n\} = \{J = n+1\}$ is determined by the value of $X_1, \ldots, X_{n+1}$. Thus, J is a stopping time, but $J - 1$ is not.

(b) [7 points] Use a martingale argument to show that $\mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

Solution: First, we have

$$\mathbb{E}(T_{J-1} + c) = \mathbb{E}(T_J - t_J + c) = \mathbb{E}(T_J - \frac{J}{\lambda} + \frac{J}{\lambda} - t_J + c) = \mathbb{E}(T_J - \frac{J}{\lambda}) + \frac{\mathbb{E}J}{\lambda} - \mathbb{E}(t_J) + c\,.$$

We calculate the three expectations in the r.h.s. above respectively. It is easy to see that J has geometric distributuion with parameter $p = \mathbb{P}(t_1 > c) = e^{-\lambda c}$. Then $\mathbb{E}J = e^{c\lambda}$. Also,

$$\mathbb{E}(t_J) = \mathbb{E}(t_1 | t_1 > c) = c + \mathbb{E}(t_1) = c + \frac{1}{\lambda}.$$

To calculate $\mathbb{E}(T_J - \frac{J}{\lambda})$, since we have shown that $X_n = T_n - \frac{n}{\lambda}$ is a martingale and J is a stopping time, by the fact that $\mathbb{E}J < \infty$ and that

$$\mathbb{E}\left(|X_{n+1} - X_n| \,\Big|\, X_1, \ldots, X_n\right) = \mathbb{E}\left(\left|t_1 - \frac{1}{\lambda}\right| \,\Big|\, X_1, \ldots, X_n\right) \le \mathbb{E}(t_1) + \frac{1}{\lambda} < \infty\,,$$

we apply the martingale stopping theorem to obtain

$$\mathbb{E}(T_J - \frac{J}{\lambda}) = \mathbb{E}(T_1 - \frac{1}{\lambda}) = 0\,.$$

Plugging in, we have

$$\mathbb{E}(T_{J-1} + c) = 0 + \frac{e^{\lambda c}}{\lambda} - c - \frac{1}{\lambda} + c = \frac{e^{\lambda c} - 1}{\lambda}\,.$$

3. We study a queue with impatient customers. Customers arrive at a single server as a Poisson process with rate λ and require an exponential amount of service with rate μ. customers waiting in line are impatient and if they are not in service they will leave at rate δ independent of their position in the queue. Show that for any $\delta > 0$ the system has a stationary distribution, and find an expression for this distribution.

Solution: Let $X(t)$ denote the process. It has states $\{0, 1, 2, \ldots\}$. We can write the transition matrix as follows.

$$\begin{aligned} q_{i,i+1} &= \lambda \\ q_{i,i-1} &= \mu + (i-1)\delta, \quad i \ge 1 \\ q_{i,k} &= 0, \quad \text{otherwise} \end{aligned}$$

It is clear that we have a death and birth process. THen, the stationary distributuion P_i satisfies:

$$\sum_{i=0}^{\infty} P_i = 1 \quad \text{and} \quad P_i q_{i,i+1} = P_{i+1} q_{i+1,i}\,. \tag{2}$$

By solving (2), we have

$$P_i = \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} P_0 \,, \quad P_0 = \left(1 + \sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)}\right)^{-1} .$$

To see the existence of stationary distribution, one can check that $P_0 > 0$. Indeed, since $\lambda > 0$, there exists $J \in \mathbb{N}$ such that $\mu + J\delta > \lambda$. Then, it follows that

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} < \sum_{i=1}^{J-1} \frac{\lambda^i}{\prod_{j=0}^{i-1}(\mu + j\delta)} + \sum_{i=J}^{\infty} \left(\frac{\lambda}{\mu + J\delta}\right)^i < \infty \,.$$

4. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate λ. The customer and the dealer then disappear from the street for an amount of time with distribution G while the transaction is completed. Customers that arrive during this time go away never to return.

(a) [5 points] At what rate does the dealer make sales? Explain your reasoning.

<u>Solution</u>: We can define a renewal process, each cycle ending when a transaction is finished. Let X_i denote the length of cycle i. Let T be a random variable with distributuion G. Then,

$$\mathbb{E}(X_i) = \frac{1}{\lambda} + \mathbb{E}(T) \,.$$

The long-run rate of transactions is then $1/\mathbb{E}[X_1]$.

(b) [5 points] What fraction of customers are lost? Explain your reasoning.

<u>Solution</u>: The fraction of customers that are lost is the stationary distribution of state 1, i.e., $\frac{\mathbb{E}(T)}{\frac{1}{\lambda} + \mathbb{E}(T)}$.

5. Let $\{Z(t), 0 \le t \le 1\}$ be a Brownian bridge, i.e., a Gaussian diffusion with $\mathbb{E}[Z(t)] = 0$ and $\mathrm{Cov}(Z(s), Z(t)) = s \wedge t - st$. Define $X(t) = (1 + t)Z(t/(1 + t))$. Show that $\{X(t), t \ge 0\}$ is a standard Brownian motion.

<u>Solution</u>: Since $Z(t)$ is a Brownian bridge, it is a Gaussian diffusion. Then $X(t)$ is a Gaussian diffusion. Now we prove

$$\mathbb{E}(X(t)) = 0 \quad \text{and} \quad \mathrm{Cov}(X(s), X(t)) = s \wedge t - st \,.$$

First,

$$\begin{aligned} \mathbb{E}(X(t)) &= (1+t)\mathbb{E}\left(Z\left(\frac{t}{1+t}\right)\right) = (1+t)\mathbb{E}\left(B\left(\frac{t}{1+t}\right) - \frac{t}{1+t}B(1)\right) \\ &= (1+t)\left(0 - \frac{t}{1+t}\right) = 0 \,. \end{aligned}$$

Next, by the fact that $Z(t)$ is a Brownian bridge,

$$\begin{aligned} \mathrm{Cov}(X(s), X(t)) &= \mathrm{Cov}\left((1+s)Z\left(\frac{s}{1+s}\right), (1+t)Z\left(\frac{t}{1+t}\right)\right) \\ &= (1+s)(1+t)\frac{s}{1+s} \wedge \frac{t}{1+t} - \frac{st}{(1+s)(1+t)} \\ &= s(1+t) \wedge t(1+s) - st = s \wedge t - st \,. \end{aligned}$$

We have thus proved that $X(t)$ is a standard Brownian motion.