Statistics 620
Final exam, Winter 2015

Name: ______________________________ **UMID #:** ______________________________

- **There are 5 questions, each worth 10 points.**
- **You are allowed two single-sided sheets of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. Suppose that shocks occur according to a Poisson process with rate λ. Suppose that each shock, independently, causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $\mathbb{P}(N = n \mid T = t)$.

2. Four children are playing two video games. The first game, which takes an average of 4 minutes to play, is not very exciting, so when a child completes a turn on it they always stand in line to play the other one. The second one, which takes an average of 8 minutes, is more interesting so, upon completing the game, the child will get back in line to play it with probability $1/2$ or go to the other machine with probability $1/2$. Assuming that they turns take an exponentially distributed amount of time, find the stationary distribution of the number of children playing or in line at each of the two machines.

3. The Markov property can be stated as "the past and the future are independent given the present." However, the Markov property does not imply that the past and future are independent given *any* information concerning the present. Find a simple example of a homogeneous, discrete time Markov chain X_n with state space $\{1, 2, 3, 4, 5, 6\}$ such that

$$\mathbb{P}\big[X_2 = 6 \,|\, X_1 \in \{3, 4\}, X_0 = 2\big] \neq \mathbb{P}[X_2 = 6 \,|\, X_1 \in \{3, 4\}].$$

4. An urn contains a red ball, a blue ball, a green ball and a white ball. A sequence of balls is drawn from the urn at random. After each draw, the color of the ball is noted and then the ball is returned to the urn. Find the expected number of draws until the first time that four consecutive balls of the same color appear.

Hint: if you choose to follow a martingale approach to this problem, it may help to imagine four gamblers arriving at each time point. One gambler places a fair bet of $1 on red; if she wins, she bets her initial stake and earnings on red again at the next turn; if she loses, she quits the game. The three other gamblers follow a similar strategy with blue, green and white.

5. This question studies the maximum of a Brownian bridge. Let $\{Z(t)\}$ be a Brownian Bridge, i.e., $\{Z(t)\}$ has the same distribution as a standard Brownian motion $\{B(t)\}$ conditioned on $B(1) = B(0) = 0$. Let $M = \max_{0\le t\le 1} Z(t)$.

(a) Derive an expression for $\mathbb{P}[\max_{0\le t\le 1} B(t) \ge m,\ B(1) \le \delta]$ for $\delta \le m$.

(b) Using this expression, find an expression for $\mathbb{P}[\max_{0\le t\le 1} B(t) \ge m \,|\, 0 \le B(1) \le \delta]$. Take a limit as $\delta \to 0$ to obtain $\mathbb{P}[M \ge m]$.