

Statistics 620
Final exam, Winter 2015

1. Suppose that shocks occur according to a Poisson process with rate λ . Suppose that each shock, independently, causes the system to fail with probability p . Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find $\mathbb{P}(N = n | T = t)$.

Solution: Split the Poisson process to count non-failure shocks, $\{N_1(t)\}$, and failure shocks, $\{N_2(t)\}$. Then, $\mathbb{P}(N = n | T = t) = \mathbb{P}(N_1(T) = n - 1 | T = t) = [(p - 1)\lambda t]^{n-1} \exp\{-(p - 1)\lambda t\} / (n - 1)!$.

2. Four children are playing two video games. The first game, which takes an average of 4 minutes to play, is not very exciting, so when a child completes a turn on it they always stand in line to play the other one. The second one, which takes an average of 8 minutes, is more interesting so, upon completing the game, the child will get back in line to play it with probability $1/2$ or go to the other machine with probability $1/2$. Assuming that they turns take an exponentially distributed amount of time, find the stationary distribution of the number of children playing or in line at each of the two machines.

Solution: Let $X(t)$ denote the number of children at game 1 at time t . $X(t)$ fully determines the state of the system, which can be represented by a Markov chain on $0, 1, 2, 3, 4$ with $q_{i,i+1} = 1/(16 \text{ min}) = \lambda$ for $i = 0, 1, 2, 3$ and $q_{i,i-1} = 1/(4 \text{ min}) = 4\lambda$ for $i = 1, 2, 3, 4$. Solving the detailed balance equations gives a stationary distribution of

$$P_i = (1/4)^i / \sum_{k=0}^4 (1/4)^k.$$

3. The Markov property can be stated as “the past and the future are independent given the present.” However, the Markov property does not imply that the past and future are independent given *any* information concerning the present. Find a simple example of a homogeneous, discrete time Markov chain X_n with state space $\{1, 2, 3, 4, 5, 6\}$ such that

$$\mathbb{P}[X_2=6 | X_1 \in \{3, 4\}, X_0=2] \neq \mathbb{P}[X_2=6 | X_1 \in \{3, 4\}].$$

Solution:

Consider a Markov chain with transition probability

$$p_{24} = p_{46} = p_{62} = 1$$

$$p_{35} = p_{51} = p_{13} = 1$$

This MC has two irreducible classes.

$$\mathbb{P}[X_2 = 6 | X_1 \in \{3, 4\}, X_0 = 2] = 1 \text{ but}$$

$$\mathbb{P}[X_2 = 6 | X_1 \in \{3, 4\}] \neq 1.$$

4. An urn contains a red ball, a blue ball, a green ball and a white ball. A sequence of balls is drawn from the urn at random. After each draw, the color of the ball is noted and then the ball is returned to the urn. Find the expected number of draws until the first time that four consecutive balls of the same color appear.

Hint: if you choose to follow a martingale approach to this problem, it may help to imagine four gamblers arriving at each time point. One gambler places a fair bet of \$1 on red; if she wins, she bets her initial stake and earnings on red again at the next turn; if she loses, she quits the game. The three other gamblers follow a similar strategy with blue, green and white.

Solution: Let Z_n be the profit of all gamblers up to time n , and T the stopping time for the first occurrence of four consecutive balls of the same color. At time T , one gambler has earned $4^4 - 1$, one has $4^3 - 1$, one has $4^2 - 1$, one has $4^1 - 1$. The other $4T - 3$ gamblers have lost \$1. Applying the martingale stopping theorem (check the conditions) gives $\mathbb{E}[Z_T] = \mathbb{E}[4^4 + 4^3 + 4^2 + 4^1 - 4T] = 0$, and so $\mathbb{E}[T] = 4^3 + 4^2 + 4 + 1$.

5. This question studies the maximum of a Brownian bridge. Let $\{Z(t)\}$ be a Brownian Bridge, i.e., $\{Z(t)\}$ has the same distribution as a standard Brownian motion $\{B(t)\}$ conditioned on $B(1) = B(0) = 0$. Let $M = \max_{0 \leq t \leq 1} Z(t)$.

(a) Derive an expression for $\mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m, B(1) \leq \delta]$ for $\delta \leq m$.

(b) Using this expression, find an expression for $\mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m \mid 0 \leq B(1) \leq \delta]$. Take a limit as $\delta \rightarrow 0$ to obtain $\mathbb{P}[M \geq m]$.

Solution: For $m \geq \delta$, a symmetry argument (see notes, or Ross Section 8.2) gives

$$\mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m, B(1) \leq \delta] = \mathbb{P}[B(1) \geq 2m - \delta] = \Phi(\delta - 2m),$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal c.d.f. and p.d.f. respectively. Therefore,

$$\begin{aligned} \mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m, 0 \leq B(1) \leq \delta] &= \Phi(\delta - 2m) - \Phi(-2m) \\ &= \delta\phi(-2m) + O(\delta) \\ &= \delta\phi(2m) + O(\delta) \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{P}[M \geq m] &= \mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m \mid W(1) = 0] \\ &= \lim_{\delta \rightarrow 0} \mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m \mid 0 \leq W(1) \leq \delta] \\ &= \lim_{\delta \rightarrow 0} \mathbb{P}[\max_{0 \leq t \leq 1} B(t) \geq m, 0 \leq W(1) \leq \delta] / \mathbb{P}[0 \leq W(1) \leq \delta] \\ &= \lim_{\delta \rightarrow 0} \frac{\delta\phi(2m) + O(\delta)}{\delta\phi(0) + O(\delta)} = \frac{\phi(2m)}{\phi(0)} \\ &= \left(\frac{1}{\sqrt{2\pi}}e^{-(2m)^2/2}\right) / \left(\frac{1}{\sqrt{2\pi}}\right) = e^{-2m^2}. \end{aligned}$$