Statistics 620
Final exam, Winter 2017

Name: ______________________________ **UMID #:** ______________________________

- **There are** 5 **questions, each worth** 10 **points.**
- **You are allowed two single-sided sheets of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

1. Each of n statistics PhD students is carrying out an independent computationally intensive research project. However, the students have access to only a single shared computer. Each student works for an exponentially distributed time with mean $1/\lambda$, after which time they submit job on the computer that requires an exponentially distributed amount of computer time with mean $1/\mu$. Only one job can run at a time on the computer, and jobs run in the order they are submitted. The students do no work after submitting a job until they get the results back from the computer, at which point they start another similarly distributed cycle of work and computation. Find the long run probability that exactly k of the n students are working at any given time.

2. We study a simple model for public health control of an Ebola outbreak. One infected individual arrives in New York City at day zero of the outbreak. Each subsequent day, the individual either infects a new person (with probability p) or the individual becomes symptomatic and is discovered by public health officials (with probability $1 - p$). Each newly infected individual behaves like the first. Find the cumulative distribution function of the time until the outbreak is discovered.

3. Let S and T be stopping times for a sequence of random variables, $X_1, X_2, \ldots$. Show that the following are all stopping times:

(a) The minimum, $U = S \wedge T$.

(b) The maximum, $V = S \vee T$.

(c) The sum, $W = S + T$.

4. We use martingales to study a simple investment management model. Let Z_n be wealth at time n. At each timepoint $n \geq 1$, we allocate a fraction F of our wealth to a risky investment, modeled as

$$Z_n = \begin{cases} Z_{n-1}(1+F) & \text{with probability } p, \\ Z_{n-1}(1-F) & \text{with probability } 1-p, \end{cases}$$

for $1/2 < p < 1$. An investment strategy involves determining F as a function of $Z_0, \ldots, Z_{n-1}$. We want to choose a stragegy to maximize the expected interest rate, $\mathbb{E}[\log(Z_N/Z_0)]$ where N is a fixed integer time and Z_0 is a known constant.

(a) Find the investment strategy optimizing $E[\log(Z_n/Z_{n-1})|Z_{n-1}]$.

(b) Argue that $\log Z_n - \alpha n$ is a supermartingale for any investment strategy, where $\alpha = p \log p + (1-p)\log(1-p) + log 2$. Use part (a) to show that there is a strategy for which $\log Z_n - n\alpha$ is a martingale.

(c) Explain how (a) and (b) determine a strategy optimizing $\mathbb{E}[\log(Z_N/Z_0)]$.

5. Let $\{B_k(t), 1 \le k \le n, t \ge 0\}$ be a collection of n independent standard Brownian motions. Let $X(t) = \sum_{k=1}^{n} [B_k(t)]^2$.

(a) Find $\lim_{h\to 0} h^{-1}\mathbb{E}[X(t+h) - X(t)|X(t) = x]$ and $\lim_{h\to 0} h^{-1}\mathrm{Var}[X(t+h) - X(t)|X(t) = x]$.

(b) Argue that $\{X(t)\}$ is a diffusion process and write down the stochastic differential equation that it solves.

(c) Let A_t be the event that $X(s)$ hits zero for some $s \ge t$. What do you think $\lim_{t\to\infty} \mathbb{P}(A_t)$ is, for each value of n? You may use results we established in class for the random walk to guide your reasoning.