

Statistics 620  
Midterm exam, Fall 2011

Name: \_\_\_\_\_ UMID #: \_\_\_\_\_

**Midterm Exam**

- There are 4 questions, each worth 10 points.
- You are allowed a single-sided sheet of notes.
- You are not allowed to make use of a calculator, or any other electronic device, during the exam.
- Credit will be given for clear explanation and justification, as well as for getting the correct answer.
- Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.
- You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. A box contains  $a$  white balls and  $b$  black balls. Balls are randomly drawn from the box, one at a time. If the drawn ball is white, it is returned to the box. If the drawn ball is black, it is painted white and then returned to the box. Find the expected number of white balls in the box after the  $n$ th draw.

**2. (a)** Let  $N(t)$  be a Poisson process with rate  $\lambda$ . Find the probability that  $N(t)$  is even.

Hint: you may or may not wish to proceed as follows. Let  $e(t)$  be the probability that  $N(t)$  is even; condition on  $N(t - \delta)$  and construct a differential equation by taking the limit as  $\delta \rightarrow 0$ .

**(b)** The owner of a computer store hands out discount coupons to every other visitor entering the store, starting with the first arrival (i.e., to the 1st, 3rd, 5th, 7th, ...). Suppose that arrival of visitors follows a Poisson process with rate  $\lambda$ . Find the expected number of coupons given out by time  $t$ .

**3.** Let  $N(t)$  be a renewal process with interarrival distribution  $F$ . Let  $W$  be the time at which the age of the renewal process first exceeds some constant  $s$ . In other words, writing  $S_n$  for the  $n$ th arrival time, define  $W = \inf\{t : t - S_{N(t)} > s\}$ . Let  $V(t) = \mathbb{P}[W \leq t]$ .

(a) Determine  $\mathbb{E}[W]$ .

(b) Establish an integral equation satisfied by  $V(t)$ .

4. Let  $\{X_n\}$  be a homogeneous Markov chain with states  $\{1, 2, 3, 4\}$  having transition probabilities given by the matrix  $P = [P_{ij}]$ . Let  $f_{ij}(n)$  be the probability mass function of the first passage time from state  $i$  to state  $j$ , defined as

$$f_{ij}(n) = \mathbb{P}[X_n = j, X_{n-1} \neq j, \dots, X_2 \neq j, X_1 \neq j \mid X_0 = i].$$

[NOTE: THE DEFINITION IN THE ORIGINAL EXAM WAS SLIGHTLY DIFFERENT, BUT I THINK THIS DEFINITION IS MORE NATURAL]

Evaluate  $f_{12}(n)$  for

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$