Statistics 620
Midterm exam, Fall 2011

1. A box contains a white balls and b black balls. Balls are randomly drawn from the box, one at a time. If the drawn ball is white, it is returned to the box. If the drawn ball is black, it is painted white and then returned to the box. Find the expected number of white balls in the box after the nth draw.

Solution: Let X_n be the number of white balls after the nth draw, and let $M_n = \mathbb{E}[X_n]$. Conditioning on X_{n-1} gives

$$\begin{aligned} M_n &= \mathbb{E}\left[X_{n-1} + \left(1 - \frac{X_{n-1}}{a+b}\right)\right] \\ &= M_{n-1} \times \frac{a+b-1}{a+b} + 1. \end{aligned}$$

This has general solution $M_n = Ax^n + B$ where $x = (a+b-1)/(a+b)$ and $B = a+b$. Then, $M_0 = a$ gives $A = -b$.

2. **(a)** Let $N(t)$ be a Poisson process with rate λ. Find the probability that $N(t)$ is even.

Hint: you may or may not wish to proceed as follows. Let $e(t)$ be the probability that $N(t)$ is even; condition on $N(t-\delta)$ and construct a differential equation by taking the limit as $\delta \to 0$.

(b) The owner of a computer store hands out discount coupons to every other visitor entering the store, starting with the first arrival (i.e., to the 1st, 3rd, 5th, 7th, ...). Suppose that arrival of visitors follows a Poisson process with rate λ. Find the expected number of coupons given out by time t.

Solution: (a) Conditioning on $N(t-\delta)$ we get

$$e(t) = e(t-\delta)[1 - \lambda\delta + o(\delta)] + (1 - e(t-\delta))[\lambda\delta + o(\delta)].$$

Thus,

$$de/dt = \lambda(1-2e). \tag{1}$$

The general solution to $de/dt + 2\lambda e = 0$ is $e(t) = A\exp\{-2\lambda t\}$. A specific solution to $de/dt + 2\lambda e = \lambda$ is $e(t) = 1/2$. So, the general solution to (1) is $e(t) = A\exp\{-2\lambda t\} + (1/2)$. Since $e(0) = 1$, we have $A = 1/2$.

(b) Let $N(t)$ count the arrivals, and $C(t)$ count the number of coupons handed out. Since

$$C(t) = \big[N(t) + I\{N(t) \text{ is odd}\}\big]/2,$$

we have

$$\mathbb{E}[C(t)] = \mathbb{E}[N(t)]/2 + (1 - e(t))/2 = \lambda t/2 + (1 - \exp\{-2\lambda t\})/4.$$

3. Let $N(t)$ be a renewal process with interarrival distribution F. Let W be the time at which the age of the renewal process first exceeds some constant s. In other words, writing S_n for the nth arrival time, define $W = \inf\{t : t - S_{N(t)} > s\}$. Let $V(t) = \mathbb{P}\big[W \le t\big]$.

(a) Determine $\mathbb{E}[W]$.

(b) Establish an integral equation satisfied by $V(t)$.

Solution: (a) Conditioning on S_1 gives

$$\mathbb{E}[W] = s\bar{F}(s) + \int_0^s (u + \mathbb{E}[W])\, dF(u)$$

This has solution

$$\mathbb{E}[W] = s + \frac{1}{\bar{F}(s)} \int_0^s u\, dF(u).$$

(b) Again conditioning on S_1,

$$\begin{aligned} V(t) &= \mathbb{E}\big[\mathbb{P}[W \le t \,|\, S_1]\big] \\ &= \bar{F}(s) + \int_0^s V(t-u) dF(u). \end{aligned}$$

4. Let $\{X_n\}$ be a homogeneous Markov chain with states $\{1,2,3,4\}$ having transition probabilities given by the matrix $P = [P_{ij}]$. Let $f_{ij}(n)$ be the probability mass function of the first passage time from state i to state j, defined as

$$f_{ij}(n) = \mathbb{P}\big[X_n = j, X_{n-1} \neq j, \ldots, X_2 \neq j, X_1 \neq j \,|\, X_0 = i\big].$$

[NOTE: THE DEFINITION IN THE ORIGINAL EXAM WAS SLIGHTLY DIFFERENT, BUT I THINK THIS DEFINITION IS MORE NATURAL]

Evaluate $f_{12}(n)$ for

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

Solution: Conditioning on the first jump, for $n \ge 2$, we have

$$f_{12}(n) = (1/2) f_{42}(n-1) \tag{2}$$

$$f_{42}(n) = (1/2) f_{12}(n-1) + (1/2) f_{32}(n-1) \tag{3}$$

$$f_{32}(n) = (1/2) f_{42}(n-1) = f_{12}(n) \tag{4}$$

Combining (3) and (4) gives $f_{42}(n) = f_{12}(n-1)$. Putting this into (2) we obtain, for $n \ge 3$,

$$f_{12}(n) = (1/2) f_{12}(n-2).$$

Since $f_{12}(1) = 1/2$ and $f_{12}(2) = 0$, this has solution

$$f_{12}(2k-1) = (1/2)^k \quad \text{and} \quad f_{12}(2k) = 0$$

for $k = 1, 2, \ldots$