Statistics 620
Midterm exam, Fall 2012

1. Workplace accidents occur at University of Michigan as a Poisson process with rate λ. Each accident independently results in a lawsuit with probability p. Let T denote the time of the first lawsuit, and let A be the total number of accidents in the random time interval $[0, T]$. Find $P[A = n|T = t]$.

Hint: you may (if you wish) use without proof the result that, if $N(t)$ is a Poisson process and each event is independently classified as Type I or Type II, then the corresponding counting processes $N_1(t)$ and $N_2(t)$ for events of each type are independent Poisson processes.

Solution: Let $N_1(t)$ be the number of accidents not resulting in a lawsuit by time t and $N_2(t)$ be the number of accidents resulting in a lawsuit by time t. Then $N_1(t)$ is a Poisson process with rate $\lambda(1-p)$, $N_2(t)$ is a Poisson process with rate λp and $N_1(t)$ is independent of $N_2(t)$.

$$\begin{aligned}
P(A = n|T = t) &= P(N_1(t) + N_2(t) = n|N_2(t) = 1, N_2(s) = 0 \text{ for } s < t) \\
&= P(N_1(t) = n-1),\ N_1 \text{ and } N_2 \text{ are independent} \\
&= \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^{n-1}}{(n-1)!}
\end{aligned}$$

2. Let $\{N(t), t \geq 0\}$ be a renewal process, with corresponding arrival times $\{S_n\}$ and inter-arrival times given by $X_n = S_n - S_{n-1}$. Suppose X_n has distribution F. Define the age process by $A(t) = t - S_{N(t)}$, namely the time since the most recent arrival.

(i) Show that $E[A(t)] = t\bar{F}(t) + \int_0^t (t-u)\bar{F}(t-u)dm(u)$.

Hint: recall that $dF_{S_{N(t)}}(u) = \begin{cases} \bar{F}(t) + \bar{F}(t)dm(0) & \text{for } u = 0 \\ \bar{F}(t-u)dm(u) & \text{for } u > 0 \end{cases}$

Solution:

$$\begin{aligned}
E[A(t)] &= E[t - S_{N(t)}] \\
&= \int_0^t (t-u)\mathrm{d}F_{S_{N(t)}}(u)\ ,\ \text{since } 0 \leq S_{N(t)} < t \\
&= (t-0)(\overline{F}(t)(1 + \mathrm{d}m(0)) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u) \\
&= t\overline{F}(t) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u)\ ,\ \text{since } \mathrm{d}m(0) = 0
\end{aligned}$$

(ii) Hence, show that $\lim_{t\to\infty} E[A(t)] = E[X^2]/2E[X]$, where X has distribution F.

Solution: Assuming X has finite second moment, $E[X^2] = \int_0^\infty 2x\overline{F}(x)\mathrm{d}x$, as proved in a homework problem. Also this implies that

$$\lim_{t\to\infty} t\overline{F}(t) = 0$$

Thus

$$\begin{aligned}
\lim_{t\to\infty} E[A(t)] &= \lim_{t\to\infty}[t\overline{F}(t) + \int_0^t (t-u)\overline{F}(t-u)\mathrm{d}m(u)] \\
&= 0 + \lim_{t\to\infty}\int_0^t (t-u)\overline{F}(t-u)\mathrm{d} \\
&= \frac{1}{E[X]}\int_0^t t\overline{F}(t)\mathrm{d}t \text{ , using Key Renewal Theorem} \\
&= \frac{E[X^2]}{2E[X]}
\end{aligned}$$

3. Trials are performed in sequence. If the two most recent previous trials were both successes, the next trial is a success with probability 0.8; otherwise, the chance of success is 0.5.

(i) Define state 1 to be {most recent trial was a failure}, state 2 to be {most recent trial was a success, and the preceding trial was a failure} and state 3 to be {last two trials were successes}. Let X_n be the state after the nth trial. Explain why $\{X_n\}$ is a Markov chain, and find the transition matrix $P = [P_{ij}]$.

(ii) Use the Markov chain in (i) to find the long run proportion of trials that are successes.

<u>Solution</u>: **3(i)**

$$P[X_{n+1} = x|X_n = 1, X_{n-1}, \cdots] = P[X_{n+1} = x|X_n =' ?F'] = \begin{cases} P[F|X_n =' ?F'] = .5 & x = 1 \\ P[S|X_n =' ?F'] = .5 & x = 2 \\ 0 & x = 3 \end{cases}$$

$$P[X_{n+1} = x|X_n = 2, X_{n-1}, \cdots] = P[X_{n+1} = x|X_n =' FS'] = \begin{cases} P[F|X_n =' FS'] = .5 & x = 1 \\ P[S|X_n =' FS'] = .5 & x = 3 \\ 0 & x = 2 \end{cases}$$

$$P[X_{n+1} = x|X_n = 3, X_{n-1}, \cdots] = P[X_{n+1} = x|X_n =' SS'] = \begin{cases} P[F|X_n =' SS'] = .2 & x = 1 \\ P[S|X_n =' SS'] = .8 & x = 3 \\ 0 & x = 2 \end{cases}$$

The above shows that $\{X_n\}$ is a markov chain and also gives the transition matrix as

$$P = \begin{pmatrix} .5 & .5 & 0 \\ .5 & 0 & .5 \\ .2 & 0 & .8 \end{pmatrix}$$

(ii) The chain is irreducible since all states communicate with each other and it is aperiodic since $P_{11} > 0$. Therefore the chain is ergodic and has a unique stationary distribution satisfying $\pi = \pi P$. The set of linear equations is easy to solve, and gives $\pi_1 = 4/11$, $\pi_2 = 2/11$ and $\pi_3 = 5/11$.

$$\lim_{n\to\infty} P[S] = \pi_2 + \pi_3 = 7/11.$$

4. If an individual has never had a previous automobile accident, then the probability he or she has an accident in the next h time units is $\beta h + o(h)$; on the other hand, if she or he has ever had a previous accident, then the probability is $\alpha h + o(h)$. Find the expected number of accidents an individual has by time t.

Hint: you may like to condition on the time of the first accident.

Solution: This question cannot readily be answered without additionally assuming independent increments and absence of simultaneous events. In applied probability, clarifying the problem is sometimes part of the solution! In this case, the time until the first accident can be recognized as having an exponential distribution (the time until the first arrival of a Poisson process) with subsequent accidents conditionally following a Poisson process with rate α.

Let X_1 be the time of the first accident. By the equivalent definitions of a Poisson process, $X_1 \sim$ Exponential(β). Similarly, the subsequent interaccident times $X_2, X_3, \ldots$ are *iid* Exponential (α). Let $N(t)$ be the number of accidents by time t.

$$
\begin{aligned}
E[N(t)] &= E[E[N(t)|X_1]] \\
&= \int_0^\infty E[N(t)|X_1 = s]\beta e^{-\beta s} ds \\
&= \int_0^t E[N(t) - N(s) + 1|X_1 = s]\beta^{-\beta s} ds \\
&= \int_0^t [1 + \alpha(t-s)]\beta e^{-\beta s} ds \\
&= (1+\alpha t)[-e^{-\beta s}]_0^t - \alpha[[-se^{-\beta s}]_0^t + \int_0^t e^{-\beta s} ds] \\
&= (1+\alpha t)(1-e^{-\beta t}) + \alpha t e^{-\beta t} - \alpha[-e^{-\beta s}/\beta]_0^t \\
&= \alpha t + (1-e^{-\beta t}) - \frac{\alpha}{\beta}(1-e^{-\beta t}) \\
&= (1-\frac{\alpha}{\beta})(1-e^{-\beta t}) + \alpha t.
\end{aligned}
$$