Statistics 620
Midterm exam, Fall 2013

1. Richard catches trout according to a Poisson process with rate 0.1 minute^{-1}. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find expressions for the mean and standard deviation of the total weight of fish he catches in two hours.

Solution: Let $N(t)$ count the number of fish caught by time t, and let $\{X_i, i = 1, 2, \ldots\}$ be a sequence of random variables giving the weight of each fish. We assume that these weights are iid and independent of $N(t)$. Let $N = N(2)$ count the fish caught in 2 hours. Let $W = \sum_{i=1}^{N} X_i$. Then,

$$E[W] = E\big\{E[W|N]\big\} = E[N]E[X_1]$$

$$\begin{aligned} Var(W) &= E[Var(W|N)] + Var(E[W|N]) \\ &= E[N\, Var(X_1)] + Var(N\, E[X_1]) \\ &= E[N]\, Var(X_1) + (E[X_1])^2 Var(N). \end{aligned}$$

Here, $E[N] = 120 \times 0.1 = 12$ and $Var(N) = E[N] = 12$. So, $E[W] = 12 \times 4 = 48$ pounds, $sd(W) = \sqrt{12 \times 4 + 12 \times 4^2} = 15.5$ pounds.

2. A light bulb has a lifetime that is exponential with a mean of $\mu = 200$ days. When it burns out, a janitor replaces it immediately. In addition, there is a handyman who comes, on average, $h = 3$ times per year according to a Poisson process and replaces the lightbulb as "preventative maintenance."

(a) Find the average time between bulb replacements, in terms of μ and h.

(b) In the long run, what fraction of the replacements are due to failure? Your answer should be an expression in terms of μ and h.

Solution: (a) Let $N_1(t)$ count burnout events, and $N_2(t)$ count handyman replacements. Then $N_3(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\frac{1}{\mu} + \frac{h}{365} = 0.0132$ day^{-1} (assuming independence of $N_1(t)$ and $N_2(t)$). Therefore, average time between replacements is

$$\frac{1}{1/\mu + h/365} = \frac{365\mu}{365 + \mu h} = 75.6 \text{ days.}$$

(b)

$$\begin{aligned} \lim_{t\to\infty} \frac{N_1(t)}{N_3(t)} &= \lim_{t\to\infty} \frac{N_1(t)}{t} \Big/ \lim_{t\to\infty} \frac{N_3(t)}{t} \\ &= \frac{1}{\mu} \times \frac{365\mu}{365 + \mu h} \\ &= \frac{365}{365 + \mu h} = \quad 0.38 \quad \text{w.p.1} \end{aligned}$$

3. Let $\{N(t), t \geq 0\}$ be a renewal process, with corresponding arrival times $\{S_n\}$ and inter-arrival times $X_n = S_n - S_{n-1}$ with $E[X_n] = \mu$. Show that $\lim_{t\to\infty} N(t)/t = 1/\mu$, with probability one.

Note: you can use without proof the strong law of large numbers for S_n.

Solution: This is in the notes.

4. Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, one of the numbers $1, 2, \ldots, 52$ is chosen at random and with equal probability. If number i is chosen, we move the card from position i in the deck to the top, i.e. to position 1. We repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.

Solution: Let X_n denote the ordering of the cards after the n^{th} operation. X_n is a Markov Chain whose state space is orderings of $1, \ldots, 52$. Note that X_n is irreducible (since in 52 operations we can reach any order) and aperiodic (since $\mathbf{P}[X_n = X_{n-1}] > 0$). Then X_n has a unique stationary distribution which is also the limiting distinction. Let π be the uniform distribution on orderings, so $\pi_i = 1/52!$. Note that the transition matrix of X_n is

$$P_{ij} = \begin{cases} \{1/52 & \text{if there is a card that can be placed on top to get } j \text{ from } i \\ 0 & \text{else} \end{cases}$$

Then $\sum_i \pi_i P_{ij} = \sum_{i:P_{ij}>0} \frac{1}{52!} \times \frac{1}{52} = 52 \times \frac{1}{52!} \times \frac{1}{52} = \frac{1}{52!} = \pi_j$. This gives the uniform distribution as the limiting distribution.