Statistics 620
Midterm exam, Winter 2015

Name: ______________________________ **UMID #:** ______________________________

Midterm Exam

- **There are 4 questions, each worth 10 points.**
- **You are allowed a single-sided sheet of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. Let X and Y be independent, non-negative, continuous random variables with densities $f_X(x)$ and $f_Y(y)$, and cumulative distribution functions $F_X(x)$ and $F_Y(y)$. Define the failure rate function for X to be

$$\lambda_X(x) = \lim_{\delta \to 0} \frac{\mathbb{P}[x < X < x + \delta \,|\, X > x]}{\delta} = \frac{f_X(x)}{1 - F_X(x)}.$$

Define λ_Y similarly, and write $\min(a, b)$ for the minimum of a and b. Show that

$$\mathbb{P}\big[X < Y \mid \min(X, Y) = t\big] = \frac{\lambda_X(t)}{\lambda_X(t) + \lambda_Y(t)}.$$

2. Consider a sequence of tosses of a fair coin, each toss being H or T with equal probability. Find the expected number of tosses before the first occurrence of the sequence HHTHHT.

Hint: Let $\{N(t)\}$ be a counting process which counts times when the most recent 6 tosses are HHTHHT in an infinite sequence of coin tosses. Let $\{N^*(t)\}$ count times when the most recent 3 tosses are HHT. You may want to use these processes in your solution. If you choose to argue that $\{N(t)\}$ is a delayed renewal process and $\{N^*(t)\}$ is an undelayed renewal process, you should explain your reasoning.

3. A worker sequentially works on jobs. Each time a job is completed, a new one is started. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. Find an expression for the rate at which jobs completed, in the long run.

4. Consider a random walk on the vertices of a cube, where at each time step a particle has equal probability of remaining in its current location or moving along an edge to an adjacent vertex. Specifically, suppose that the vertices of the cube are labeled $\{1, 2, \ldots, 8\}$ and let $\{X_n\}$ be a Markov chain taking values in this set. The one-step transition probabilities are $P_{ii} = 1/4$ and $P_{ij} = 1/4$ if i and j are connected along an edge of the cube, with $P_{ij} = 0$ otherwise. Let μ_{ii} be the expected return time to state i. Compute μ_{ii}.

Hint: A nice approach involves using Blackwell's lattice renewal theorem and identifying the stationary distribution of the Markov chain. Optionally, you may instead derive μ_{ii} by any argument of your choice.