Statistics 620
Midterm exam, Winter 2015

1. Let X and Y be independent, non-negative, continuous random variables with densities $f_X(x)$ and $f_Y(y)$, and cumulative distribution functions $F_X(x)$ and $F_Y(y)$. Define the failure rate function for X to be

$$\lambda_X(x) = \lim_{\delta\to 0} \frac{\mathbb{P}[x < X < x+\delta \,|\, X > x]}{\delta} = \frac{f_X(x)}{1-F_X(x)}.$$

Define λ_Y similarly, and write $\min(a,b)$ for the minimum of a and b. Show that

$$\mathbb{P}\big[X < Y \mid \min(X,Y) = t\big] = \frac{\lambda_X(t)}{\lambda_X(t)+\lambda_Y(t)}.$$

Solution:

$$\begin{aligned}
\mathbb{P}\big[X < Y \mid \min(X,Y=t)\big] &= \lim_{\delta\to 0} \mathbb{P}\big[X<Y \mid \min(X,Y)\in[t,t+\delta]\big] \\
&= \lim_{\delta\to 0} \frac{\mathbb{P}\big[X<Y, \min(X,Y)\in[t,t+\delta]\big]}{\mathbb{P}\big[\min(X,Y)\in[t,t+\delta]\big]} \\
&= \lim_{\delta\to 0} \frac{\mathbb{P}\big[X\in[t,t+\delta], Y>t\big]+o(\delta)}{\mathbb{P}\big[X\in[t,t+\delta], Y>t\big]+\mathbb{P}\big[Y\in[t,t+\delta], X>t\big]+O(\delta)} \\
&= \lim_{\delta\to 0} \frac{\delta f_X(t)\bar{F}_Y(t)+o(\delta)}{\delta f_X(t)\bar{F}_Y(t)+\delta f_Y(t)\bar{F}_X(t)+o(\delta)} \\
&= \frac{f_X(t)/\bar{F}_X(t)}{f_X(t)/\bar{F}_X(t)+f_Y(t)/\bar{F}_Y(t)} = \frac{\lambda_X(t)}{\lambda_X(t)+\lambda_Y(t)}
\end{aligned}$$

2. Consider a sequence of tosses of a fair coin, each toss being H or T with equal probability. Find the expected number of tosses before the first occurrence of the sequence HHTHHT.

Hint: Let $\{N(t)\}$ be a counting process which counts times when the most recent 6 tosses are HHTHHT in an infinite sequence of coin tosses. Let $\{N^*(t)\}$ count times when the most recent 3 tosses are HHT. You may want to use these processes in your solution. If you choose to argue that $\{N(t)\}$ is a delayed renewal process and $\{N^*(t)\}$ is an undelayed renewal process, you should explain your reasoning.

Solution: Let $N(t)$ be the delayed renewal process counting times when the most recent 6 tosses are HHTHHT. Let $N^*(t)$ count times when the most recent 3 tosses are HHT. Then, $\mathbb{E}[X_2] = 2^6$, and X_1 is equal in distribution to $X_2 + X_1^*$. $\{N^*(t)\}$ is an undelayed renewal process, since after any renewal time, including zero, the renewal interval is ≥ 3 and occurs exactly at the first time HHT reoccurs. Therefore, we have $\mathbb{E}[X_1^*] = 2^3$ and so $\mathbb{E}[X_1] = 2^6 + 2^3$.

3. A worker sequentially works on jobs. Each time a job is completed, a new one is started. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. Find an expression for the rate at which jobs completed, in the long run.

Solution: Take rewards to occur when a new job is started, and assign a reward of 1 if the job is completed and 0 otherwise, with $R(t)$ the resulting renewal reward process. Set $X \sim F$ and independently $Y \sim$ Exponential (λ).

$$\begin{aligned}
\text{long run completion rate} &= \lim_{t\to\infty} \frac{R(t)}{t} \\
&= \frac{\mathbb{E}[\text{reward for one cycle}]}{\mathbb{E}[\text{length of cycle}]} \\
&= \frac{\mathbb{P}[X < Y]}{E[\min(x,y)]} \\
&= \frac{\int_0^\infty F(x)\lambda e^{-\lambda x}dx}{\int_0^\infty E[\min(X,x)]\lambda e^{-\lambda x}dx} \\
&= \frac{\int_0^\infty F(x)\lambda e^{-\lambda x}dx}{\int_0^\infty \int_0^x \overline{F}(y)dy\lambda e^{-\lambda x}dx}
\end{aligned}$$

4. Consider a random walk on the vertices of a cube, where at each time step a particle has equal probability of remaining in its current location or moving along an edge to an adjacent vertex. Specifically, suppose that the vertices of the cube are labeled $\{1, 2, \ldots, 8\}$ and let $\{X_n\}$ be a Markov chain taking values in this set. The one-step transition probabilities are $P_{ii} = 1/4$ and $P_{ij} = 1/4$ if i and j are connected along an edge of the cube, with $P_{ij} = 0$ otherwise. Let μ_{ii} be the expected return time to state i. Compute μ_{ii}.

Hint: A nice approach involves using Blackwell's lattice renewal theorem and identifying the stationary distribution of the Markov chain. Optionally, you may instead derive μ_{ii} by any argument of your choice.

Solution: Blackwell's lattice renewal theorem gives us

$$\lim_{n\to\infty} \mathbb{P}[X_n = i] = 1/\mu_{ii}. \tag{1}$$

Since $\{X_n\}$ is an irreducible, aperiodic, finite Markov chain, $1/\mu_{ii}$ is also the stationary distribution which is 1/8 by symmetry. So, $\mu_{ii} = 8$.