Statistics 620
Midterm exam, Winter 2017

Name: ______________________________ **UMID #:** ______________________________

Midterm Exam

- **There are 4 questions, each worth 10 points.**
- **You are allowed a single-sided sheet of notes.**
- **You are not allowed to make use of a calculator, or any other electronic device, during the exam.**
- **Credit will be given for clear explanation and justification, as well as for getting the correct answer.**
- **Cross out any working that you do not wish to be considered as part of your solution. You are advised not to erase unfinished working since partial credit may be available for an indication that an appropriate method was attempted, even if it was later rejected.**
- **You may continue your solutions on the reverse side of the pages. Additional paper is available, should you require it.**

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. Buses leaves the bus terminal on schedule, every T minutes. Potential passengers arrive at the terminal as a Poisson process, rate λ. If an individual arrives M minutes before a bus leaves, they take that bus with probability $e^{-\mu M}$. With the remaining probability $1 - e^{-\mu M}$, the individual walks away from the bus station, perhaps to head into a cafe or go shopping. Let N be the number of passengers on the bus. Find $\mathbb{P}\{N = n\}$ for $n = 0, 1, 2, \ldots$.

2. Let $\{N(t)\}$ be a renewal process with interarrival times $X_n = S_n - S_{n-1}$. Suppose X_n has probability density function $f(x)$ and mean μ. Let $A(t) = t - S_{N(t)}$ and $Y(t) = S_{N(t)+1} - t$ be the age and residual life processes with corresponding densities $f_{A(t)}(a)$ and $f_{Y(t)}(y)$. Find the joint density, $f_{A(t),Y(t)}(a, y)$ in the limit as $t \to \infty$. Evaluate this in the case that $\{N(t)\}$ is a Poisson process with rate $\lambda = 1/\mu$.

Hint: it may be helpful to obtain the limit of the complementary cumulative distribution function, $\lim_{t\to\infty} \bar{F}_{A(t),Y(t)}(a, y) = \lim_{t\to\infty} \mathbb{P}\big[A(t) > a, Y(t) > y\big]$.

3. Recall two definitions for a counting process $\{N(t), t \geq 0\}$ to be a rate λ Poisson process:

Definition 1.
(i) $N(0) = 0$,
(ii) $N(t)$ has independent increments,
(iii) $N(t) - N(s) \sim$ Poisson $(\lambda(t-s))$ for $s < t$.

Definition 2.
(i) $N(0) = 0$,
(ii) $N(t)$ has stationary independent increments,
(iii) $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$,
(iv) $\mathbb{P}(N(h) \geq 2) = o(h)$.

Show that definition 2 implies definition 1. You may use, without proof, the Poisson limit of the binomial distribution.

4. Let $\{X_n, n \geq 0\}$ be a Markov chain carrying out a random walk on the 3×3 grid, with states labeled $\{1, 2, \ldots, 9\}$ as follows:

1	2	3
4	5	6
7	8	9

Suppose that transitions to all adjacent locations are equally likely, for example, $\mathbb{P}[X_{n+1} = 4 \,|\, X_n = 1] = \mathbb{P}[X_{n+1} = 2 \,|\, X_n = 1] = 1/2$. Suppose also that $X_0 = 1$. Find the chance that the first time the Markov chain leaves the top row it enters state 4. In other words, find $\mathbb{P}[A]$ for

$$A = \bigcup_j \{X_j = 4\} \bigcap_{k<j} \{X_k \in \{1, 2, 3\}\}.$$