Statistics 620
Midterm exam, Winter 2017

1. Buses leaves the bus terminal on schedule, every T minutes. Potential passengers arrive at the terminal as a Poisson process, rate λ. If an individual arrives M minutes before a bus leaves, they take that bus with probability $e^{-\mu M}$. With the remaining probability $1 - e^{-\mu M}$, the individual walks away from the bus station, perhaps to head into a cafe or go shopping. Let N be the number of passengers on the bus. Find $\mathbb{P}\{N = n\}$ for $n = 0, 1, 2, \ldots$.

Solution: In the time interval $[0, T]$ we have a split Poisson process, with an arrival at time t having probability $e^{-\mu(T-t)}$ of being split into the bus-boarding category. Therefore, N is Poisson distributed, with mean

$$\nu = \mathbb{E}[N] = \int_0^T \lambda e^{-\mu t}\, dt = \frac{\lambda}{\mu}\left(1 - e^{-\mu T}\right).$$

and so, $\mathbb{P}\{N = n\} = \nu^n e^{-\nu}/n!$.

2. Let $\{N(t)\}$ be a renewal process with interarrival times $X_n = S_n - S_{n-1}$. Suppose X_n has probability density function $f(x)$ and mean μ. Let $A(t) = t - S_{N(t)}$ and $Y(t) = S_{N(t)+1} - t$ be the age and residual life processes with corresponding densities $f_{A(t)}(a)$ and $f_{Y(t)}(y)$. Find the joint density, $f_{A(t),Y(t)}(a, y)$ in the limit as $t \to \infty$. Evaluate this in the case that $\{N(t)\}$ is a Poisson process with rate $\lambda = 1/\mu$.

Hint: it may be helpful to obtain the limit of the complementary cumulative distribution function, $\lim_{t\to\infty} \bar{F}_{A(t),Y(t)}(a, y) = \lim_{t\to\infty} \mathbb{P}\big[A(t) > a, Y(t) > y\big]$.

Solution: Set up a regenerative process with regeneration times at renewal times of $\{N(t)\}$. The limiting joint complementary distribution function of $A(t)$ and $Y(t)$ is

$$\begin{aligned}
\lim_{t\to\infty} \bar{F}_{A(t),Y(t)}(a, y) &= \frac{\text{Expected time with age} > a \text{ and residual life} > y \text{ in one renewal period}}{\text{expected length of one renewal period}} \\
&= \frac{1}{\mu}\mathbb{E}\left[\max\{X - (a + y), 0\}\right] \\
&= \frac{1}{\mu}\int_{a+y}^{\infty} [x - (a + y)]\, f(x)\, dx
\end{aligned}$$

Differentiating, we obtain

$$\begin{aligned}
f_{A(t),Y(t)}(a, y) &= \frac{1}{\mu}\frac{\partial^2}{\partial a \partial y}\int_{a+y}^{\infty} [x - (a + y)]\, f(x)\, dx \\
&= \frac{1}{\mu}\frac{\partial}{\partial a}\int_{a+y}^{\infty} -f(x)\, dx \\
&= \frac{f(a + y)}{\mu}.
\end{aligned}$$

In the case of a Poisson process, we obtain

$$f_{A(t),Y(t)}(a, y) = \lambda^2 \exp\{-\lambda(a + y)\} \tag{1}$$

and so the age and residual life are independent Exponential(λ) random variables.

3. Recall two definitions for a counting process $\{N(t), t \geq 0\}$ to be a rate λ Poisson process:

Definition 1.
(i) $N(0) = 0$,
(ii) $N(t)$ has independent increments,
(iii) $N(t) - N(s) \sim$ Poisson $(\lambda(t-s))$ for $s < t$.

Definition 2.
(i) $N(0) = 0$,
(ii) $N(t)$ has stationary independent increments,
(iii) $\mathbb{P}\left(N(h) = 1\right) = \lambda h + o(h)$,
(iv) $\mathbb{P}\left(N(h) \geq 2\right) = o(h)$.

Show that definition 2 implies definition 1. You may use, without proof, the Poisson limit of the binomial distribution.

Solution: We follow the notes. (i) and (ii) of Definition 1 are immediate. To show (iii), we divide $[0, t]$ into n equal subintervals and define

$$X_{nk} = \begin{cases} 1 & \text{if} N\big(k\,t/n\big) - N\big((k-1)\,t/n\big) \geq 1 \\ 0 & \text{else} \end{cases} \tag{2}$$

Then set $X_n = \sum_{k=1}^{n} X_{nk}$, so X_n counts the number of subintervals with at least one event. Let $A_{nk} = \big\{N(kt/n) - N\big((k-1)t/n\big) \geq 2\big\}$. By (iv) and (ii), $\mathbb{P}[A_{nk}] = o(t/n)$ and $\big\{X_n = N(t)\big\}^c = \bigcup_{k=1}^{n} A_{nk}$. So,

$$\mathbb{P}\left[\big\{X_n = N(t)\big\}^c\right] \leq \sum_{k=1}^{n} \mathbb{P}[A_{nk}] = n\mathbb{P}[A_{n1}].$$

Since $\lim_{n\to\infty} n\, o(t/n) = 0$, we have

$$\lim_{n\to\infty} \mathbb{P}[X_n = N(t)] = 1. \tag{3}$$

Then, by (ii), $\{X_{n,k}, k = 1, \ldots, n\}$ are IID, so $X_n \sim$ Binomial(n, p_n), where (iii) and (iv) give

$$p_n = \lambda \frac{t}{n} + o\left(\frac{t}{n}\right).$$

Then, the Poisson limit of the binomial distribution gives that X_n has a limiting Poisson distribution with mean λt. Using (3), we get

$$\mathbb{P}[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

4. Let $\{X_n, n \geq 0\}$ be a Markov chain carrying out a random walk on the 3×3 grid, with states labeled $\{1, 2, \ldots, 9\}$ as follows:

1	2	3
4	5	6
7	8	9

Suppose that transitions to all adjacent locations are equally likely, for example, $\mathbb{P}[X_{n+1} = 4 \,|\, X_n = 1] = \mathbb{P}[X_{n+1} = 2 \,|\, X_n = 1] = 1/2$. Suppose also that $X_0 = 1$. Find the chance that the first time the Markov chain leaves the top row it enters state 4. In other words, find $\mathbb{P}[A]$ for

$$A = \bigcup_j \{X_j = 4\} \bigcap_{k<j} \{X_k \in \{1,2,3\}\}.$$

<u>Solution</u>: Let p_i be the probability of A given $X_0 = i$, for $i = 1, 2, 3$. By conditioning on the first transition, we get

$$\begin{aligned} p_1 &= 1/2 + p_2/2 && (4)\\ p_2 &= p_1/3 + p_3/3 && (5)\\ p_3 &= p_2/2 && (6) \end{aligned}$$

Putting (6) into (5), we get $p_2 = \frac{2}{5}p_1$ and then (4) gives

$$p_1 = \frac{5}{8}.$$