Homework 10 (Stats 620, Winter 2017)

Due Tuesday April 18, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. A stochastic process $\{X(t), t \geq 0\}$ is said to be *stationary* if $X(t_1), ..., X(t_n)$ has the same joint distribution as $X(t_1 + a), ..., X(t_n + a)$ for all $n, a, t_1, ..., t_n$.

   (a) Prove that a necessary and sufficient condition for a Gaussian process to be stationary is that $\mathrm{Cov}((X(s), X(t))$ depends only on $t - s$, $s \leq t$, and $\mathbb{E}[X(t)] = c$.

   (b) Let $\{X(t), t \geq 0\}$ be Brownian motion and define

   $$V(t) = e^{-\alpha t/2} X(\alpha e^{\alpha t}).$$

   Show that $\{V(t), t \geq 0\}$ is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process.

2. Let $X(t)$ be standard Brownian motion. Find the distribution of:

   (a) $\big|X(t)\big|$.

   (b) $\big|\min_{0 \leq s \leq t} X(s)\big|$

   (c) $\max_{0 \leq s \leq t} X(s) - X(t)$

   Hint: all three parts have the same answer.

3. Let $M(t) = \max_{0 \leq s \leq t} X(s)$ where $X(t)$ is standard Brownian motion. Show that

   $$\mathbb{P}\{M(t) > a \mid M(t) = X(t)\} = e^{-a^2/2t}, \quad a > 0.$$

   Hint: One approach is outlined below. There may be other ways.

   (i) Differentiate the expression

   $$P(M(t) > y, B(t) < x) = \int_{2y-x}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t}\, du$$

   to find the joint density of $M(t)$ and $B(t)$.
   (ii) Transform variables to find the joint density of $M(t)$ and $M(t) - B(t)$. This involves using the Jacobian formula (e.g. Ross, A First Course in Probability, 6th edition, Section 6.7): If X_1 and X_2 have joint density $f_{X_1 X_2}$, $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$, $X_1 = h_1(Y_1, Y_2)$ and $X_2 = h_2(Y_1, Y_2)$, then (supposing suitable regularity)

   $$f_{Y_1 Y_2}(y_1, y_2) = \frac{f_{X_1 X_2}(h_1(y_1, y_2), h_2(y_1, y_2))}{|J(h_1(y_1, y_2), h_2(y_1, y_2))|}$$

   where J is the matrix determinant (Jacobian) given by

   $$J(x_1, x_2) = \begin{vmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{vmatrix}$$

   (iii) Find the conditional density of $M(t)$ given $M(t) - B(t) = 0$.

4. For a Brownian motion process with drift coefficient μ, let

$$f(x) = \mathbb{E}\left[\text{time to hit either } A \text{ or } -B \mid X_0 = x\right],$$

where $A > 0, B > 0, -B < x < A$.

(a) Derive a differential equation for $f(x)$.

(b) Solve this equation.

(c) Use a limiting random walk argument (see Problem 4.22 of Chapter 4) to verify the solution in part (b).

Recommended reading:
Sections 8.3, 8.4, 8.5.

Supplementary exercises: 8.3, 8.4, 8.6, 8.16
Optional, but recommended. Do not turn in solutions—they are in the back of the book.