Homework 10 (Stats 620, Winter 2017)

Due Tuesday April 18, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. A stochastic process $\{X(t), t \geq 0\}$ is said to be *stationary* if $X(t_1), ..., X(t_n)$ has the same joint distribution as $X(t_1 + a), ..., X(t_n + a)$ for all $n, a, t_1, ..., t_n$.

   (a) Prove that a necessary and sufficient condition for a Gaussian process to be stationary is that $\text{Cov}((X(s), X(t))$ depends only on $t - s$, $s \leq t$, and $\mathbb{E}[X(t)] = c$.

   (b) Let $\{X(t), t \geq 0\}$ be Brownian motion and define

   $$V(t) = e^{-\alpha t/2} X(\alpha e^{\alpha t}).$$

   Show that $\{V(t), t \geq 0\}$ is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process.

   Solution:

   If the Gaussian process is stationary then for $t > s$

   $$\begin{pmatrix} X(t) \\ X(s) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X(t-s) \\ X(0) \end{pmatrix}$$

   Thus $\mathbb{E}[X(s)] = \mathbb{E}[X(0)]$ for all s and $\text{Cov}(X(t), X(s)) = \text{Cov}(X(t-s), X(0))$ for all $t < s$.

   Now, assume $\mathbb{E}[X(t)] = c$ and $\text{Cov}(X(t), X(s)) = h(t-s)$. For any $T = (t_1, \cdots, t_k)$ define vector $X_T \equiv (X(t_1), \cdots, X(t_k))'$. Let $\tilde{T} = (t_1 - a, \cdots, t_k - a)$. If $\{X(t)\}$ is a Gaussian process then both X_T and $X_{\tilde{T}}$ are multivariate normal and it suffices to show that they have the same mean and covariance. This follows directly from the fact that they have the same element-wise mean c and the equal pair-wise covariances, $\text{Cov}(X(t_i - a), X(t_j - a)) = h(t_i - t_j) = \text{Cov}(X(t_i), X(t_j))$.

   (b) Since all finite dimensional distributions of $\{V(t)\}$ are Normal, it is a Gaussian process. Thus from part (a) it suffices to show the following:

   (a) $\mathbb{E}[V(t)] = e^{-\alpha t/2}\mathbb{E}[X(\alpha e^{\alpha t})] = 0$. Thus $\mathbb{E}[V(t)]$ is constant.

   (b) For $s \leq t$,

   $$\text{Cov}(V(s), V(t)) = e^{-\alpha(t+s)/2}\text{Cov}(X(\alpha e^{\alpha s}), X(\alpha e^{\alpha t})) = e^{-\alpha(t+s)/2}\alpha e^{\alpha s} = \alpha e^{-\alpha(t-s)/2},$$

   which depends only on $t - s$.

2. Let $X(t)$ be standard Brownian motion. Find the distribution of:

   (a) $|X(t)|$.

   (b) $|\min_{0 \leq s \leq t} X(s)|$

   (c) $\max_{0 \leq s \leq t} X(s) - X(t)$

Hint: all three parts have the same answer.

Solution:

(a) Let $Y(t) = |X(t)|$. For $y \geq 0$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(|X(t)| \leq y) \\ &= \mathbb{P}(-y \leq X(t) \leq y) = 2\int_0^y \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u\,. \end{aligned}$$

(b) Let $Y(t) = |\min_{0\leq s\leq t} X(s)|$. For $y \geq 0$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y(t) \leq y) = \mathbb{P}(\min_{0\leq s\leq t} X(s) \geq -y) \\ &= \mathbb{P}(T_{-y} \geq t) = 1 - \int_{y/\sqrt{t}}^{\infty} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u \\ &= \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u\,. \end{aligned}$$

(c) Let $Y = \max_{0\leq s\leq t} X(s)$ and $X = X(t)$ then

$$F(x, y) \equiv \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) - \mathbb{P}(Y > y, X \leq x)\,.$$

Let Φ and ϕ be the distribution and density functions respectively of a standard normal random variable. Using results derived in class,

$$F(x, y) = \Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{x - 2y}{\sqrt{t}}\right), y \geq x, y > 0\,.$$

Thus,

$$\frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{2}{t}\phi'\left(\frac{x - 2y}{\sqrt{t}}\right).$$

Since the Jacobian for the transformation $V = Y - X$, $W = Y$ is of unit modulus, the density of (V, W) is given by

$$f(v, w) = \frac{2}{t}\phi'\left(\frac{v - w}{\sqrt{t}}\right), v, w \geq 0\,. \tag{1}$$

Thus

$$\begin{aligned} \mathbb{P}(Y - X \leq y) &= \mathbb{P}(V \leq y) = \int_0^y \int_0^\infty \frac{2}{t}\phi'\left(\frac{v - w}{\sqrt{t}}\right) \mathrm{d}w\mathrm{d}v \\ &= \int_0^y \frac{2}{\sqrt{t}}\phi\left(\frac{v}{\sqrt{t}}\right) \mathrm{d}v \qquad (2) \\ &= \sqrt{\frac{2}{\pi}} \int_0^{y/\sqrt{t}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u\,. \end{aligned}$$

3. Let $M(t) = \max_{0 \le s \le t} X(s)$ where $X(t)$ is standard Brownian motion. Show that

$$\mathbb{P}\{M(t) > a \mid M(t) = X(t)\} = e^{-a^2/2t}, \quad a > 0.$$

Hint: One approach is outlined below. There may be other ways.

(i) Differentiate the expression

$$P(M(t) > y, B(t) < x) = \int_{2y-x}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t}\, du$$

to find the joint density of $M(t)$ and $B(t)$.

(ii) Transform variables to find the joint density of $M(t)$ and $M(t) - B(t)$. This involves using the Jacobian formula (e.g. Ross, A First Course in Probability, 6th edition, Section 6.7): If X_1 and X_2 have joint density $f_{X_1X_2}$, $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$, $X_1 = h_1(Y_1, Y_2)$ and $X_2 = h_2(Y_1, Y_2)$, then (supposing suitable regularity)

$$f_{Y_1Y_2}(y_1, y_2) = \frac{f_{X_1X_2}(h_1(y_1, y_2), h_2(y_1, y_2))}{|J(h_1(y_1, y_2), h_2(y_1, y_2))|}$$

where J is the matrix determinant (Jacobian) given by

$$J(x_1, x_2) = \begin{vmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{vmatrix}$$

(iii) Find the conditional density of $M(t)$ given $M(t) - B(t) = 0$.

<u>Solution</u>:

$$V = \max_{0 \le s \le t} X(s) - X(t) \quad \text{and} \quad W = \max_{0 \le s \le t} X(s)\,.$$

The joint density of (V, W) is given by equation (1), and the marginal density of V follows from equation (2):

$$f_V(v) \;=\; \int_0^{\infty} \frac{2}{t} \phi' \left(\frac{v - w}{\sqrt{t}} \right) \mathrm{d}w = \frac{2}{\sqrt{t}} \phi \left(\frac{v}{\sqrt{t}} \right).$$

The conditional density of W given $V = 0$ is $f(0, w)/f_V(0)$, which gives

$$\mathbb{P}(W \le a | V = 0) \;=\; \int_0^a \frac{f(0, w)}{f_V(0)} \mathrm{d}w = 1 - \frac{\phi(-a/\sqrt{t})}{\phi(0)}\,.$$

Thus

$$\mathbb{P}(W > a | V = 0) \;=\; 1 - \mathbb{P}(W \le a | V = 0) = e^{-\frac{a^2}{2t}}\,.$$

4. For a Brownian motion process with drift coefficient μ, let

$$f(x) = \mathbb{E}\left[\text{time to hit either } A \text{ or } -B \mid X_0 = x\right],$$

where $A > 0, B > 0, -B < x < A$.

(a) Derive a differential equation for $f(x)$.

(b) Solve this equation.

(c) Use a limiting random walk argument (see Problem 4.22 of Chapter 4) to verify the solution in part (b).

Solution:

(a) Note that the conditional distribution of process $\{Y(t) = X(t+h) : t \geq 0 | X(h) = x\}$ is the same as distribution of $\{X(t) : t \geq 0 | X(0) = x\}$. Thus if $T(x)$ =time to hit either A or $-B$ given $X(0) = x$, then

$$T(x) = h + T(X(h)) + o(h).$$

Thus for $Y = X(h) - X(0)$,

$$f(x) \equiv \mathbb{E}[T(x)] = h + \mathbb{E}[f(x+Y)] + o(h).$$

From the Taylor series expansion

$$f(x) = h + \mathbb{E}[f(x) + f'(x)Y + f'(x)Y^2/2 + \cdots] + o(h),$$

it follows that,

$$h + f'(x)\mu h + f''(x)(\mu^2 h^2 + h)/2 + o(h) = 0.$$

Dividing the equation above by h on both side, we obtain

$$1 + f'(x)\mu + f''(x)(\mu^2 h + 1)/2 = o(h)/h,$$

That is, letting $h \to 0$,

$$1 + f'(x)\mu + f''(x)/2 = 0. \tag{3}$$

(b) Let $v = 1 + f'(x)\mu$. Equation (3) becomes

$$v + \frac{1}{2\mu}\frac{\mathrm{d}v}{\mathrm{d}x} = 0$$

Thus $v = c_1 e^{-2\mu x} = 1 + f'(x)\mu$. This gives

$$f'(x) = \frac{c_1 e^{-2\mu x} - 1}{\mu}.$$

Finally

$$f(x) = \frac{1}{\mu}\left(\frac{c_1 e^{-2\mu x} - 1}{-2\mu} - x\right) + c_2. \tag{4}$$

Using the boundary conditions $f(A) = 0 = f(-B)$, equation (4) gives

$$f(x) = \frac{A+B}{\mu}\left(\frac{e^{-2\mu x} - e^{-2\mu A}}{e^{-2\mu A} - e^{2\mu B}}\right) + \frac{A-x}{\mu}. \tag{5}$$

(c) From the class notes for $T = T(0)$,

$$\begin{aligned}\mathbb{E}[T] &= \lim_{n\to\infty} \frac{\mathbb{E}[T^{(n)}]}{n} \\ &= \lim_{n\to\infty} \frac{1}{n} \frac{AP_A - B(1-P_A)}{\mathbb{E}[(Y_1 + \mu/\sqrt{n})/\sqrt{n}]} \\ &= \lim_{n\to\infty} \frac{1}{n} \frac{AP_A - B(1-P_A)}{\mu/n} .\end{aligned}$$

Here

$$P_A \approx \frac{1 - e^{-\theta_n B}}{e^{\theta_n A} - e^{-\theta_n B}} , \tag{6}$$

where θ_n satisfies

$$\mathbb{E}[e^{\theta_n (Y_1 + \mu/\sqrt{n})/\sqrt{n}}] = 1 .$$

Thus

$$\mathbb{E}[e^{\theta_n Y_1/\sqrt{n}}] = e^{-\theta_n \mu/n} .$$

Since Y_1 is standard normal. By the moment generating function

$$e^{\theta_n^2/(2n)} = e^{-\theta_n \mu/n} ,$$

it follows that $\theta_n = -2\mu$. Substituting in (6), we obtain

$$P_A \approx \frac{1 - e^{2\mu B}}{e^{-2\mu A} - e^{2\mu B}} .$$

Thus

$$\mathbb{E}[T] \approx \frac{A+B}{\mu} \left(\frac{1 - e^{2\mu B}}{e^{-2\mu A} - e^{2\mu B}} \right) - \frac{B}{\mu} ,$$

which is the same as $f(0)$ obtained earlier.

Recommended reading:
Sections 8.3, 8.4, 8.5.

Supplementary exercises: 8.3, 8.4, 8.6, 8.16
Optional, but recommended. Do not turn in solutions—they are in the back of the book.