Homework 2 (Stats 620, Winter 2017)

Due Tuesday January 31, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Calculate $\mathbb{E}\big[N(t)N(t+s)\big]$.

   Comment: Please state carefully where you make use of basic properties of Poisson processes, such as stationary, independent increments.

   Solution: Note that $N(t)N(t+s) = N(t)[N(t) + N(t+s) - N(t)]$. Thus

$$\mathbb{E}[N(t)N(t+s)] = \mathbb{E}[(N(t))^2] + \mathbb{E}[N(t)(N(t+s) - N(t))] = \mathbb{E}[(N(t))^2] + \mathbb{E}[N(t)]\mathbb{E}[N(s)]$$

   where we used the property of stationary and independent increments. Since for any $t > 0$, $N(t) \sim$Poisson(λt), whence $\mathbb{E}[N(t)] = \lambda t$, $\mathrm{Var}[N(t)] = \lambda t$. It follows that

$$\mathbb{E}[N(t)N(t+s)] = \lambda t + (\lambda t)^2 + \lambda s \lambda t\,.$$

2. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2. Show that $\{N_1(t)+N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1+\lambda_2$. Also, show that the probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1 + \lambda_2)$, independently of the time of the event.

   Solution: We check that $N(t) = N_1(t) + N_2(t)$ satisfies Definition 1.
   (i) $N(t) = 0$.
   (ii) Note that $N_1(t)$ and N_2 have independent increments. Moreover, $N_1(t)$ and $N_2(t)$ are independent.
   (iii) Indeed, for any $t, s > 0$,

$$\begin{aligned}
\mathbb{P}(N(t+s) - N(t) = n) &= \sum_{k=0}^{n} \mathbb{P}(N_1(t+s) - N_1(t) = n-k | N_2(t+s) - N_2(t) = k) \\
&\qquad \times \mathbb{P}(N_2(t+s) - N_2(t) = k) \\
&= \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k}}{(n-k)!} \exp\{-\lambda_1 s\} \frac{(\lambda_2 s)^k}{k!} \exp\{-\lambda_2 s\} \\
&= \exp\{-(\lambda_1+\lambda_2)s\} \sum_{k=0}^{n} \frac{(\lambda_1 s)^{n-k}(\lambda_2 s)^k}{(n-k)!k!} \\
&= \frac{((\lambda_1+\lambda_2)s)^n}{n!} \exp\{-(\lambda_1+\lambda_2)t\}\,.
\end{aligned}$$

   Now to show that the probability of the first arrival is from $N_1(t)$. Let X be the first arrival time for $N(t)$, and X_1, X_2 the corresponding times for $N_1(t)$ and $N_2(t)$.

One way to do is, observing that, $X \sim$ Exponential$(\lambda_1 + \lambda_2)$,

$$\begin{aligned}
\mathbb{P}(\text{first event from } N_1(t)|X = x) &= \lim_{\delta_x \to 0} \mathbb{P}(X_1 < X_2 | X \in [x, x + \delta_x]) \\
&= \lim_{\delta_x \to 0} \frac{\mathbb{P}(X_1 \in [x, x+\delta_x])\mathbb{P}(X_2 > x) + o(\delta_x)}{\mathbb{P}(X \in [x, x+\delta_x])} \\
&= \frac{e^{-\lambda_1 x}(\lambda_1 \delta_x + o(\delta_x))e^{-\lambda_2 x} + o(\delta_x)}{e^{-(\lambda_1+\lambda_2)x}((\lambda_1+\lambda_2)\delta_x + o(\delta_x))} = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

As required, this probability does not depend on the first event time for $N(t)$.

3. Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop the it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.
(a) Compute the expected time from when you arrive at the bus stop until you reach home.
(b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimizes by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.
(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.
Solution: **(a)** Let $E_s = \mathbb{E}$(journey time for strategy s). The journey time is the function of the first arrival time of the rate λ Poisson process of bus arrivals. This has Exponential(λ) distribution (prop 2.2.1). So

$$E_s = \int_0^\infty \lambda e^{-\lambda t}[(t+R)\mathbf{1}(t \le s) + (s+W)\mathbf{1}(t > s)]\, dt$$

where $\mathbf{1}$ is the indicator function. Thus

$$\begin{aligned}
E_s &= \int_0^s \lambda t e^{-\lambda t}\, dt + R\int_0^s \lambda e^{-\lambda t}\, dt + (s+W)\int_s^\infty \lambda e^{-\lambda t}\, dt \\
&= \frac{1 - e^{-\lambda s}}{\lambda} + R(1 - e^{-\lambda s}) + W e^{-\lambda s}
\end{aligned}$$

(b) Writing $E_s = (W - R - \frac{1}{\lambda})e^{-\lambda s} + \frac{1}{\lambda} + R$. We see that E_s is a decreasing function of s for $(W - R - 1/\lambda) > 0$, and increasing function for $(W - R - 1/\lambda) < 0$ and constant if $(W - R - 1/\lambda) = 0$.
(c) From the memoryless property of the exponential distribution , if it was worth waiting some time $s_0 > 0$ for a bus, and the bus has not arrived at s_0, then resetting time suggests that it must be worth waiting another s_0 time units. Thus, if the optimal s is positive, it must be infinite.

4. Cars pass a certain street location according to a Poisson process with rate λ. A person wanting to cross the street at that location waits until she can see that no cars will come by

in the next T time units. Find the expected time that the person waits before starting to cross. (Note, for instance, that if no cars will be passing in the first T time units then the waiting time is 0.)

Comment: An elegant approach is to condition on the first arrival time.
Solution: Let W be the waiting time, and let X be the first arrival time.

$$\begin{aligned}\mathbb{E}(W) &= \mathbb{E}\big[\mathbb{E}(W|X)\big] \\ &= \int_0^\infty \mathbb{E}(W|X=x)\lambda e^{-\lambda x}\,dx \\ &= \int_0^\infty \big[(\mathbb{E}(W)+x)\mathbf{1}(x<T) + 0\times\mathbf{1}(x\ge T)\big]\lambda e^{-\lambda x}\,dx\,,\end{aligned}$$

where the last equality follows by the fact that when $X < T$, we can use the memoryless property to reset the clock. Thus,

$$\mathbb{E}(W) = \mathbb{E}(W)\int_0^T \lambda e^{-\lambda x}\,dx + \int_0^T \lambda x e^{-\lambda x}\,dx\,,$$

which gives

$$\mathbb{E}(W) = \frac{1}{\lambda}\big[e^{\lambda T} - (1+\lambda T)\big]\,.$$

5. Individuals enter a system in accordance with a Poisson process having rate λ. Each arrival independently makes its way through the states of the system. Let $\alpha_i(s)$ denote the probability that an individual is in state i a time s after it arrived. Let $N_i(t)$ denote the number of individuals in state i at time t. Show that the $N_i(t), i \ge 1$, are independent and $N_i(t)$ is Poisson with mean equal to

$$\lambda\,\mathbb{E}\big[\text{amount of time an individual is in state } i \text{ during its first } t \text{ units in the system}\big].$$

Comment: You will probably want to make use of Theorem 2.3.1 of Ross. This question is similar to a multivariate version of Proposition 2.3.2, and you may need the multinomial distribution. If n independent experiments each give rise to outcomes $1,\dots,r$ with respective probabilities $p_1,\dots,p_r$, and X_i counts the number of outcomes of type i, then $X_1,\dots,X_r$ are *multinomial*. For $\sum_{i=1}^r n_i = n$,

$$\mathbb{P}(X_1=n_1,\dots,X_r=n_r) = \frac{n!}{n_1!\dots n_r!}\prod_{i=1}^r p_i^{n_i}.$$

Solution: Although not explicit in the question, we suppose there are countably infinite states. Let $N(t)$ be the arrival process, so $N(t) = \sum_{i=1}^\infty N_i(t)$.

$$\begin{aligned}\mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}) &= \mathbb{E}(\mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}|N(t))) \\ &= \mathbb{P}(N_i(t)=n_i\,\forall\, i\in\mathbb{N}|N(t)=n)\mathbb{P}(N(t)=n)\end{aligned}$$

where $n = \sum_i n_i$. Now let $U_1, \ldots, U_n$ be n i.i.d. random variables uniformly distributed on $[0,t]$. Theorem 2.3.1 of Ross asserts that conditional on $N(t) = n$, the arrival times in $[0,t]$ $S_1, \ldots, S_n$ have the same distribution as the ordered random variables: $U_{(1)}, \cdots, U_{(n)}$. Let U denote one uniform random variable on $[0,t]$. We define

$$\beta_i \triangleq \mathbb{P}(\text{arrival at time } U, \text{in state } i \text{ at time } t) = \frac{1}{t}\int_0^t \alpha_i(t-s)\,ds = \frac{1}{t}\int_0^t \alpha_i(s)ds\,.$$

Thus,

$$\begin{aligned}
\mathbb{P}(N_i(t) = n_i \,\forall\, i) &= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{k\text{–th arrival is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{\text{the arrival at time } U_{(k)} \text{ is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \mathbb{P}\left(\sum_{k=1}^n \mathbf{1}_{\{\text{the arrival at time } U_k \text{ is at state } i \text{ at time } t\}} = n_i \,\forall\, i\right) \\
&= \frac{n!}{\prod_{i\in\mathcal{I}} n_i!} \prod_i \beta_i^{n_i} \times \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \prod_{i\in\mathcal{I}} \frac{(\lambda t \beta_i)^{n_i}}{(n_i)!} e^{-\lambda t \beta_i} \\
&= \prod_{i\in\mathcal{I}} \mathbb{P}(N_i(t) = n_i)\,,
\end{aligned}$$

where $\mathbb{P}(N_i(t) = n_i)$ is calculated in a similar way. We have thus obtained that $\{N_i(t), i \geq 1\}$ are independent Poisson$(\lambda t \beta_i)$ random variables.

To complete the proof, define random variable $\mathbf{1}_i(s)$ be 1 if an individual is in state i after s time units and 0 otherwise. Then

$$\begin{aligned}
\mathbb{E}(\text{time in } i \text{ during individual's first } t \text{ units in the system}) &= \mathbb{E}\big[\int_0^t \mathbf{1}_i(s)\,ds\big] \\
&= \int_0^t \big[\mathbb{E}\mathbf{1}_i(s)\big]\,ds = \int_0^t \alpha_i(s)\,ds = t\beta_i\,.
\end{aligned}$$

Recommended reading:
Sections 2.1 through 2.4, excluding 2.3.1.

Supplementary exercises: 2.14, 2.22.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.