Homework 3 (Stats 620, Winter 2017)

Due Tuesday February 7, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

Hint: One approach is to use the identity $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$ for appropriate choices of X and Y.

Solution:

$$\begin{aligned} m(t) &= \mathbb{E}(N(t)) \\ &= \mathbb{E}(\mathbb{E}(N(t)|X_1)) \\ &= \int_0^t \mathbb{E}(N(t)|X_1 = x)\, dF(x) \text{ since } X_1 > t \Rightarrow N(t) = 0 \\ &= \int_0^t \mathbb{E}(1 + N(t-x))\, dF(x) \text{ since renewals are i.i.d.} \\ &= \int_0^t \big[1 + m(t-x)\big]\, dF(x) \\ &= F(t) + \int_0^t m(t-x)\, dF(x)\,. \end{aligned} \tag{1}$$

2. Prove that the renewal function $m(t), 0 \leq t < \infty$ uniquely determines the interarrival distribution F.
Hint: Laplace transforms may be useful.

Solution: Note that there are two definitions of Laplace transform. Under the definition of Ross,

$$\widetilde{F}(s) = \int_0^\infty e^{-st}\, dF(t)\,,$$

and we also have the Laplace transform of the convolution $F * G(t) = \int_0^\infty F(t-s) dG(s)$:

$$\begin{aligned} \widetilde{F*G}(s) &= \int_0^\infty \exp\{-st\} d\left(\int_0^\infty F(t-x) dG(x)\right) = \int_0^\infty \exp\{-st\} \int_0^\infty dF(t-x) dG(x) \\ &= \int_0^\infty \int_x^\infty \exp\{-st\} dF(t-x) dG(x) = \int_0^\infty \int_0^\infty \exp\{-s(t+x)\} dF(t) dG(x) \\ &= \int_0^\infty \exp\{-sx\} \int_0^\infty \exp\{-st\} dF(t) dG(x) \\ &= \int_0^\infty \exp\{-sx\} \widetilde{F}(s) dG(x) = \widetilde{F}(s) \widetilde{G}(s)\,. \end{aligned}$$

Thus the Laplace transform of Equation (1) becomes

$$\widetilde{m}(s) = \widetilde{F}(s) + \widetilde{m}(s)\widetilde{F}(s)$$

so

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1 + \widetilde{m}(s)} \,. \tag{2}$$

By the uniqueness of Laplace transforms, $\widetilde{m}(s)$ uniquely determines F. Another way to obtain Relation (2) is to calculate the Laplace transform of the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \,,$$

where F_n is the n-th convolution of F and $\widetilde{F_n}(s) = \widetilde{F}^n(s)$. If we use another definition of Laplace transform:

$$\widetilde{F}(s) = \int_0^{\infty} e^{-st} F(t) dt \,,$$

the calculation becomes slightly different. In particular, the Laplace transform of $\int_0^t m(t-s)dF(s)$ becomes $\widetilde{m}(s)s\widetilde{F}(s)$. In this case, the Relation (2) becomes

$$\widetilde{F}(s) = \frac{\widetilde{m}(s)}{1 + s\widetilde{m}(s)} \,.$$

3. Let $\{N(t), t \geq 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t) = n$, the event times $S_1, \ldots, S_n$ are distributed as the order statistics of a set of independent uniform $(0, t)$ random variables. Show that $\{N(t), \geq 0\}$ is a Poisson process.
Hint: Consider $\mathbb{E}[N(s) \mid N(t)]$ and then use the result of Problem 2.
Solution: Following the hints

$$\mathbb{E}[N(s)|N(t) = n] = \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)]$$

where $U_{(1)}, \cdots, U_{(n)}$ are the order statistics of n i.i.d. Unif$[0, t]$ random variables $U_1, \cdots, U_n$. Thus

$$\begin{aligned}
\mathbb{E}[N(s)|N(t) = n] &= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_{(i)} \leq s)] \\
&= \mathbb{E}[\sum_{i=1}^{n} \mathbf{1}(U_i \leq s)] \text{ since ordering does not affect the sum} \\
&= \sum_{i=1}^{n} \mathbb{P}[U_i \leq s] = ns/t \,.
\end{aligned}$$

Thus

$$m(s) = \mathbb{E}[\mathbb{E}[N(s)|N(t)]] = \frac{s}{t}\mathbb{E}[N(t)] = \frac{s}{t}m(t)\,.$$

The only solution to this is $m(s) = as$ for some constant a. This is exactly the renewal function for a rate a Poisson process. Using the result from question 2 completes the argument.

4. The random variables $X_1, \dots, X_n$ are said to be exchangeable if $X_{i_1}, \dots, X_{i_n}$ has the same joint distribution as $X_1, \dots, X_n$ whenever $i_1, i_2, \dots, i_n$ is a permutation of $1, 2, \dots, n$. That is, they are exchangeable if the joint distribution function $\mathbb{P}\{X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n\}$ is a symmetric function of $(x_1, x_2, \dots, x_n)$. Let $X_1, X_2, \dots$ denote the interarrival times of a renewal process.
(a) Argue that conditional on $N(t) = n, X_1, \dots, X_n$ are exchangeable. Would $X_1, \dots, X_n, X_{n+1}$ be exchangeable (conditional on $N(t) = n$)?
(b) Use (a) to prove that for $n > 0$

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}\big|N(t) = n\right] = \mathbb{E}[X_1|N(t) = n].$$

(c) Prove that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}\big|N(t) > 0\right] = \mathbb{E}[X_1|X_1 < t].$$

Hint: One approach to (a) involves computing

$$\mathbb{E}\big\{\mathbb{P}[X_1 \le x_1, \dots, X_n \le x_n, N(t) = n\big|X_1, \dots, X_n]\big\}.$$

Solution: **(a)** Employing the hint, we write

$$\begin{aligned}
&\mathbb{P}[X_1 \le x_1, \dots, X_n \le x_n, N(t) = n] \\
&= \int_{y_1 \le x_1} \cdots \int_{y_n \le x_n} \mathbb{P}\Big[X_{n+1} > t - \sum_{i=1}^n y_i\Big]\, dF(y_1)\dots dF(y_n) \\
&= \int_0^1 \cdots \int_0^1 I_{\{y_1 \le x_1, \dots, y_n \le x_n\}} \bar{F}\Big(t - \sum_{i=1}^n y_i\Big)\, dF(y_1)\dots dF(y_n).
\end{aligned}$$

Changing the order of integration, by Fubini's theorem, we see that the integral is unchanged by permutations of $x_1, \dots, x_n$.
(b) First note that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_{N(t)}}{N(t)}|N(t) = n\right] = \sum_{i=1}^n \frac{1}{n}\mathbb{E}[X_i|N(t) = n]\,.$$

By the exchangeability established in part (a), $\mathbb{E}[X_i|N(t) = n] = \mathbb{E}[X_1|N(t) = n]$, $i = 1, \cdots, n$. So the required result follows.

(c)

$$
\begin{aligned}
\mathbb{E}\left[\frac{X_1+\cdots+X_{N(t)}}{N(t)}|N(t)>0\right] &= \sum_{n=1}^{\infty}\mathbb{E}[\frac{X_1+\cdots+X_{N(t)}}{N(t)}|N(t)=n]\mathbb{P}[N(t)=n|N(t)>0] \\
&= \sum_{n=1}^{\infty}\mathbb{E}[X_1|N(t)=n]\mathbb{P}[N(t)=n|N(t)>0] \\
&= \mathbb{E}[X_1|N(t)>0]=\mathbb{E}[X_1|X_1<t]\,.
\end{aligned}
$$

5. Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey; and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let T denote the time it takes the miner to become free.
(a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i.$$

Note: You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.
(b) Use Wald's equation to find $\mathbb{E}[T]$.
(c) Compute $\mathbb{E}\big[\sum_{i=1}^{N} X_i|N=n\big]$ and note that it is not equal to $\mathbb{E}\big[\sum_{i=1}^{n} X_i\big]$.
(d) Use part (c) for a second derivation of $\mathbb{E}[T]$.

Solution: **(a)** Define

$$
X = \begin{cases} 2 & \text{Door 1 (probability } 1/3) \\ 4 & \text{Door 2 (probability } 1/3) \\ 8 & \text{Door 3 (probability } 1/3) \end{cases}
$$

and $N = \min\{n : X_n = 2\}$. Clearly N is a stopping time as the event $N = n$ is determined by the first n observations of X.
(b) Using Wald's theorem, $\mathbb{E}[T] = \mathbb{E}[N]\mathbb{E}[X]$. Further $\mathbb{E}[N] = 3$ since N follows a geometric distribution with parameter $p = 1/3$. Also $\mathbb{E}[X] = 14/3$. Thus $\mathbb{E}[T] = 14$.
(c)

$$
\begin{aligned}
\mathbb{E}[\sum_{i=1}^{N} X_i|N=n] &= \mathbb{E}[\sum_{i=1}^{N} X_i|X_1\neq 2,\cdots,X_{n-1}\neq 2, X_n=2]\,, \\
&= 2+(n-1)\mathbb{E}[X_i|X_i\neq 2] = 2+(n-1)6 = 6n-4 \\
\mathbb{E}[\sum_{i=1}^{n} X_i] &= n\mathbb{E}[X_i] = 14n/3\,.
\end{aligned}
$$

(d)$\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N} X_i|N]] = \mathbb{E}[6N-4] = 6\times 3-4 = 14.$

Recommended reading:
Sections 3.1 through 3.3.

Supplementary exercise: 3.7.
Optional, but recommended. Do not turn in a solution—it is in the back of the book.