Homework 4 (Stats 620, Winter 2017)

Due Tuesday Feb 14, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $A(t)$ and $Y(t)$ denote respectively the age and excess at t. Find:
   (a) $\mathbb{P}\{Y(t) > x | A(t) = s\}$.
   (b) $\mathbb{P}\{Y(t) > x | A(t + x/2) = s\}$.
   (c) $\mathbb{P}\{Y(t) > x | A(t + x) > s\}$ for a Poisson process.
   (d) $\mathbb{P}\{Y(t) > x, A(t) > y\}$.
   (e) If $\mu < \infty$, show that, with probability 1, $A(t)/t \to 0$ as $t \to \infty$.

   Hint: For (d), use a regenerative process argument (E.g. Ross, section 3.7) to find $\lim_{t\to\infty} \mathbb{P}(Y(t) > x, A(t) > y)$. For (e), you may use without proof the following results on convergence with probability 1: **(L1)** $\lim_{n\to\infty} S_n/n = \mu$; **(L2)** $\lim_{t\to\infty} N(t) = \infty$; **(L3)** $\lim_{t\to\infty} N(t)/t = 1/\mu$.

2. Consider a single-server bank in which potential customers arrive in accordance with a renewal process having interarrival distribution F. However, an arrival only enters the bank if the server is free when he or she arrives; otherwise, the individual goes elsewhere without being served. Would the number of events by time t constitute a (possibly delayed) renewal process if an event corresponds to a customer:
   (a) entering the bank?
   (b) leaving the bank after being served?
   What if F were exponential?

3. On each bet a gambler, independently of the past, either wins or loses 1 unit with respective probability p and $1 - p$. Suppose the gambler's strategy is to quit playing the first time she wins k consecutive bets. At the moment she quits

   (a) find her expected winnings.
   (b) find the expected number of bets that she has won.

   Hint: It may help you to look at Example 3.5(A) in Ross.

4. Prove Blackwell's theorem for renewal reward processes. That is, assuming that the cycle distribution is not lattice, show that, as $t \to \infty$,

$$\mathbb{E}\big[\text{reward in}(t, t+a)\big] \to a\,\frac{\mathbb{E}[\text{reward incycle}]}{\mathbb{E}[\text{time of cycle}]}.$$

   Assume that any relevant function is directly Riemann integrable.

   Hint: You may adopt an informal approach by assuming that one can write

$$\mathbb{E}\left[\int_t^{t+a} dR(s)\right] = \int_t^{t+a} \mathbb{E}\big[dR(s)\big],$$

   and then developing the identity

$$\mathbb{E}[dR(t)] = \mathbb{E}[R_1 | X_1 = t]dF(t) + \int_0^t \{\mathbb{E}[R_1 | X_1 = t - x]dF(t - x)\}\, dm(x).$$

If you can find a more elegant or more rigorous solution, that would also be good!

5. The life of a car is a random variable with distribution F. An individual has a policy of trading in his car either when it fails or reaches the age of A. Let $R(A)$ denote the resale value of an A-year-old car. There is no resale value of a failed car. Let C_1 denote the cost of a new car and suppose that an additional cost C_2 is incurred whenever the car fails.
(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.
(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.
Note: In both (a) and (b) you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should, of course, be the same in both parts.

Recommended reading:
Sections 3.4 through 3.7, excluding subsections 3.4.3, 3.6.1, 3.7.1. We will not cover the material in Section 3.8, though you may like to look through it.

Supplementary exercises: 3.24, 3.27, 3.35.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.