Homework 4 (Stats 620, Winter 2017)

Due Tuesday Feb 14, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Let $A(t)$ and $Y(t)$ denote respectively the age and excess at t. Find:
(a) $\mathbb{P}\{Y(t) > x | A(t) = s\}$.
(b) $\mathbb{P}\{Y(t) > x | A(t + x/2) = s\}$.
(c) $\mathbb{P}\{Y(t) > x | A(t + x) > s\}$ for a Poisson process.
(d) $\mathbb{P}\{Y(t) > x, A(t) > y\}$.
(e) If $\mu < \infty$, show that, with probability 1, $A(t)/t \to 0$ as $t \to \infty$.

Hint: For (d), use a regenerative process argument (E.g. Ross, section 3.7) to find $\lim_{t\to\infty} \mathbb{P}(Y(t) > x, A(t) > y)$. For (e), you may use without proof the following results on convergence with probability 1: **(L1)** $\lim_{n\to\infty} S_n/n = \mu$; **(L2)** $\lim_{t\to\infty} N(t) = \infty$; **(L3)** $\lim_{t\to\infty} N(t)/t = 1/\mu$.

Solution: **(a)**

$$\begin{aligned}\mathbb{P}[Y(t) > x | A(t) = s] &= \mathbb{P}[X_{N(t)+1} > s + x | S_{N(t)} = t - s] \\ &= \mathbb{P}[X_1 > s + x | X_1 > s] = \overline{F}(s + x)/\overline{F}(s)\,.\end{aligned}$$

Here is a more formal solution:

$$\begin{aligned}&\mathbb{P}[Y(t) > x | A(t) = s] \\ &= \mathbb{P}[S_{N(t)+1} > t + x | S_{N(t)} = t - s] = \mathbb{P}[X_{N(t)+1} > s + x | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | S_n = t - s, N(t) = n]\mathbb{P}[N(t) = n | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | S_n = t - s, X_{n+1} > s]\mathbb{P}[N(t) = n | S_{N(t)} = t - s] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+1} > s + x | X_{n+1} > s]\mathbb{P}[N(t) = n | S_{N(t)} = t - s] \text{(by independence)} \\ &= \mathbb{P}[X_1 > s + x | X_1 > s] = \overline{F}(s + x)/\overline{F}(s)\end{aligned}$$

(b)

$$p := \mathbb{P}[Y(t) > x | A(t + x/2) = s]$$

For $s \geq x/2$, an argument similar to (a) allows us to write

$$\begin{aligned}p &= \mathbb{P}[\text{no event in } (t + x/2 - s, t + x] |\text{ event at } t + x/2 - s, \text{ no events in } (t + x/2 - s, t + x/2]] \\ &= \mathbb{P}[X_{N(t)+1} > s + x/2 | S_{N(t)} = t + x/2 - s, X_{N(t)+1} > s] \\ &= \mathbb{P}[X_1 > s + x/2 | X_1 > s] = \overline{F}(s + x/2)/\overline{F}(s)\end{aligned}$$

For $s < x/2$, $\{A(t + x/2) = s\} \Rightarrow \{Y(t) \leq s - x/2\}$. It follows that $p = 0$.

(c)

$$\begin{aligned} q &\equiv \mathbb{P}[Y(t) > x | A(t+x) > s] \\ &= \mathbb{P}\big[\text{no event in } [t, t+x] \big| \text{ no events in } [t+x-s, t+x]\big] \end{aligned}$$

for $0 \le s \le x$, since the process is Poisson with independent increments,

$$q = \mathbb{P}\big[\text{no event in } [t, t+x-s]\big] = \exp^{-\lambda(x-s)},$$

where λ is the rate of the Poisson process. For $s > x$, $q = 1$.

(d)

$$\begin{aligned} &\mathbb{P}(Y(t) > x, A(t) > y) = \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y) \\ &\quad = \mathbb{P}(X_1 > t + x, t > y | S_{N(t)} = 0)\mathbb{P}(S_{N(t)} = 0) \\ &\quad\quad + \int_0^t \mathbb{P}(X_{N(t)+1} > t - S_{N(t)} + x, t - S_{N(t)} > y | S_{N(t)} = s, X_{N(t)+1} > t - S_{N(t)}) dF_{S_{N(t)}}(s) \\ &\quad = \mathbf{1}_{\{t>y\}}\mathbb{P}(X_1 > t + x | X_1 > t)\mathbb{P}(S_{N(t)} = 0) \\ &\quad\quad + \int_0^t \mathbf{1}_{\{t-s>y\}}\mathbb{P}(X > t + x - s | X > t - s) dF_{S_{N(t)}}(s) \\ &\quad = \mathbf{1}_{\{t>y\}}\overline{F}(t+x) + \int_0^{t-y} \overline{F}(t + x - s) dm(s)\,. \end{aligned}$$

Let $P_t = \mathbb{P}[Y(t) > x, A(t) > y]$. Define a regenerative process to be "on" at t if $S_{N(t)} < t - y$ and $S_{N(t)+1} > t + x$. Thus, P_t is the probability that the process is "on" at time t. By the regenerative process limit theorem

$$\begin{aligned} \lim_{t\to\infty} P_t &= \frac{\mathbb{E}[\text{time "on" during a cycle}]}{\mathbb{E}[\text{time of the cycle}]} \\ &= \frac{\mathbb{E}[\max(X_1 - (x+y), 0)]}{\mu} \\ &= \frac{1}{\mu}\int_{x+y}^{\infty} (z - x - y)\, dF(z) \end{aligned}$$

(e)

$$\begin{aligned} \lim_{t\to\infty} \frac{A(t)}{t} &= \lim_{t\to\infty} \frac{t - S_{N(t)}}{t} \\ &= 1 - \lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} \lim_{t\to\infty} \frac{N(t)}{t} \\ &= 1 - \mu/\mu \quad \text{(by (L1), (L2) and (L3))} \\ &= 0 \end{aligned}$$

2. Consider a single-server bank in which potential customers arrive in accordance with a renewal process having interarrival distribution F. However, an arrival only enters the bank if the

server is free when he or she arrives; otherwise, the individual goes elsewhere without being served. Would the number of events by time t constitute a (possibly delayed) renewal process if an event corresponds to a customer:
(a) entering the bank?
(b) leaving the bank after being served?
What if F were exponential?

Solution: Let X_i denote the length of the i-th service and let Y_i denote the time from the end of i-th service until the start of the $i+1$-th service. Let Y_0 denote the time when first arrival enters the bank (and gets service). Note that X_i and Y_i may be dependent when the arrival is not a Poisson process.
(a) In this case, each cycle consists of $Z_i = X_i + Y_i, i = 1, 2, \ldots$ and $Z_0 = Y_0$. Since X_i and Y_i are independent of X_j and Y_j with $j = 1, \ldots, i-1$, $\{Z_i\}_{i\in\mathbb{N}}$ are i.i.d copies. We thus have a delayed renewal process.
(b) In this case, $Z_i = Y_{i-1} + X_i$. When X_i and Y_i are dependent, $\{Z_i\}_{i\in\mathbb{N}}$ are not i.i.d. copies. We do not have a (delayed) renewal process. One counter example can be constructed as in the sequel. Suppose the service distribution is given by

$$Y_1 = \begin{cases} 1 & \text{w.p. } 0.5 \\ 10 & \text{w.p. } 0.5 \end{cases}$$

and the interarrival times of the customers to the bank $Z_n \sim F$ are given by, $Z_1 = 6$ w.p. 1. Then, given a previous interval between departures $S_n - S_{n-1} = 3$, we know that the next arrival will enter the bank at time $S_n + 4$.

If F is exponential **(a)** still gives a delayed renewal process. **(b)** now results in a (non-delayed) renewal process, since the memoryless property implies that Y_i is independent of X_i, $i \in \mathbb{N}$. Hence, $\{Z_i\}_{i\in\mathbb{N}}$ are i.i.d. copies.

3. On each bet a gambler, independently of the past, either wins or loses 1 unit with respective probability p and $1-p$. Suppose the gambler's strategy is to quit playing the first time she wins k consecutive bets. At the moment she quits

   (a) find her expected winnings.
   (b) find the expected number of bets that she has won.

   Hint: It may help you to look at Example 3.5(A) in Ross.

   Solution: Let

$$Y_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ 0 & \text{else} \end{cases}$$

   and

$$X_n = \begin{cases} 1 & \text{if } n\text{th game is a win} \\ -1 & \text{else} \end{cases}$$

   and let $N = \inf\{n \geq k : \sum_{m=n-k+1}^{n} X_m = k\} = \inf\{n \geq k : \sum_{m=n-k+1}^{n} Y_m = k\}$, the first time k consecutive games are won. Let $W = \sum_{i=1}^{N} X_i$, the gamblers total winnings. Also let $N_W = \sum_{i=1}^{N} Y_i$, the number of games won.

(a) From Ross, Example 3.5A, $\mathbb{E}[N] = \sum_{i=1}^{k}(1/p)^i$. Also N is a stopping time w.r.t $X_i, i = 1, 2, \ldots$. By Wald's equation, we have

$$\mathbb{E}[W] = \mathbb{E}[N]\,\mathbb{E}[X_1] = \left(\sum_{i=1}^{k}\frac{1}{p^i}\right)(2p-1)$$

(b) N is also a stopping time w.r.t. $Y_i, i = 1, \cdots$, so Wald's equation gives

$$\mathbb{E}[N_W] = \mathbb{E}[N]\mathbb{E}[Y_1] = \left(\sum_{i=1}^{k}\frac{1}{p^i}\right)p = \sum_{i=0}^{k-1}\frac{1}{p^i}$$

4. Prove Blackwell's theorem for renewal reward processes. That is, assuming that the cycle distribution is not lattice, show that, as $t \to \infty$,

$$\mathbb{E}\big[\text{reward in}(t, t+a)\big] \to a\,\frac{\mathbb{E}[\text{reward incycle}]}{\mathbb{E}[\text{time of cycle}]}.$$

Assume that any relevant function is directly Riemann integrable.

Hint: You may adopt an informal approach by assuming that one can write

$$\mathbb{E}\left[\int_t^{t+a} dR(s)\right] = \int_t^{t+a}\mathbb{E}\big[dR(s)\big],$$

and then developing the identity

$$\mathbb{E}[dR(t)] = \mathbb{E}[R_1|X_1 = t]dF(t) + \int_0^t \{\mathbb{E}[R_1|X_1 = t-x]dF(t-x)\}\,dm(x).$$

If you can find a more elegant or more rigorous solution, that would also be good!

Solution: Let $R(t)$ be the reward accumulated by time t. Then,

$$\begin{aligned}\mathbb{E}[\text{reward in }(t, t+a)] &= \mathbb{E}[R(t+a) - R(t)] \\ &= \mathbb{E}[\int_t^{t+a} dR(s)] \\ &= \int_t^{t+a}\mathbb{E}[\,dR(s)]\end{aligned}$$

assuming that the interchange is allowed, e.g. if $R(t)$ is increasing. Now,

$$\begin{aligned}\mathbb{E}[\,dR(t)] &= \mathbb{E}[\mathbb{E}[\,dR(t)|S_{N(t)}]] \\ &= \mathbb{E}[\,dR(t)|S_{N(t)} = 0]\mathbb{P}[S_{N(t)} = 0] + \int_0^\infty \mathbb{E}[\,dR(t)|S_{N(t)} = y]\,dF_{S_{N(t)}}(y) \\ &= \mathbb{E}[\,dR(t)|S_{N(t)} = 0]\overline{F}(t) + \int_0^\infty \mathbb{E}[\,dR(t)|S_{N(t)} = y]\overline{F}(t-y)\,dm(y)\,.\end{aligned}$$

Now, since $R(t)$ only increases when an event occurs,

$$\begin{aligned}\mathbb{E}[\,dR(t)|S_{N(t)}=y] &= \mathbb{E}[R_{N(t)+1}|X_{N(t)+1}=t-y]\mathrm{d}F_{X_{N(t)+1}|S_{N(t)}=y}(t-y)\\ &= \mathbb{E}[R_1|X_1=t-y]\frac{dF(t-y)}{\overline{F}(t-y)}\,.\end{aligned}$$

This established the identity

$$\mathbb{E}[\,dR(t)] = \mathbb{E}[R_1|X_1=t]\,dF(t) + \int_0^t (\mathbb{E}[R_1|X_1=t-y]\,dF(t-y))\,dm(x)\,.$$

Now the key renewal theorem gives

$$\lim_{t\to\infty}\mathbb{E}[\,dR(t)] = \int_0^t \mathbb{E}[R_1|X_1=t]\,dF(t)\,dt = \mathbb{E}[R_1]\,dt\,.$$

Thus

$$\lim_{t\to\infty}\int_t^{t+a}\mathbb{E}[\,dR(t)] = \int_t^{t+a}\mathbb{E}[R_1]\,ds = a\frac{\mathbb{E}[R_1]}{\mu}\,.$$

Another approach: Note that

$$\begin{aligned}\mathbb{E}[\text{reward in } (t,t+a)] &= \mathbb{E}[\sum_{n=1}^{N(t+a)} R_n - \sum_{n=1}^{N(t)} R_n]\\ &= \mathbb{E}[\sum_{n=1}^{N(t+a)+1} R_n] - \mathbb{E}[\sum_{n=1}^{N(t)+1} R_n] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\end{aligned}$$

Now $N(t+a)+1$ and $N(t)+1$ are stopping times for the sequence $(X_i,R_i), i=1,\cdots$. Thus from (generalized) Wald's equation

$$\mathbb{E}[\sum_{n=1}^{N(t)+1} R_n] = \mathbb{E}[N(t)+1]\mathbb{E}[R]$$

and

$$\mathbb{E}[\sum_{n=1}^{N(t+a)+1} R_n] = \mathbb{E}[N(t+a)+1]\mathbb{E}[R]\,,$$

where $\mathbb{E}[R]$ is the expected reward in a cycle. Thus

$$\begin{aligned}\mathbb{E}[\text{reward in } (t,t+a)] &= \mathbb{E}[N(t+a)+1]\mathbb{E}[R] - \mathbb{E}[N(t)+1]\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\\ &= (m(t+a)-m(t))\mathbb{E}[R] + \mathbb{E}[R_{N(t)+1}] - \mathbb{E}[R_{N(t+a)+1}]\,.\end{aligned}$$

Now

$$\lim_{t\to\infty}(m(t+a)-m(t))\mathbb{E}[R] = a\mathbb{E}[R]/\mathbb{E}[X]\,.$$

from Blackwell's theorem. Now, it suffices to show that $\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}]$ exists and is finite. Indeed,

$$\begin{aligned}
\mathbb{E}[R_{N(t)+1}] &= \mathbb{E}[R_{N(t)+1}|S_{N(t)}=0]\overline{F}(t) + \int_0^t \mathbb{E}[R_{N(t)+1}|S_{N(t)}=s]\overline{F}(t-s)\mathrm{d}m(s) \\
&= \mathbb{E}[R|X>t]\overline{F}(t) + \int_0^t \mathbb{E}[R|X>t-s]\overline{F}(t-s)\,dm(s) \\
&= h(t) + \int_0^t h(t-s)\,dm(s)\,,
\end{aligned}$$

where $h(t) = \mathbb{E}[R|X>t]\overline{F}(t)$. Then by the Key Renewal theorem, we have

$$\lim_{t\to\infty} \mathbb{E}[R_{N(t)+1}] = \frac{1}{\mathbb{E}[X]}\int_0^\infty h(s)\,ds\,.$$

Here we assumed that $h(t)$ is directly Riemann integrable.

5. The life of a car is a random variable with distribution F. An individual has a policy of trading in his car either when it fails or reaches the age of A. Let $R(A)$ denote the resale value of an A-year-old car. There is no resale value of a failed car. Let C_1 denote the cost of a new car and suppose that an additional cost C_2 is incurred whenever the car fails.
(a) Say that a cycle begins each time a new car is purchased. Compute the long-run average cost per unit time.
(b) Say that a cycle begins each time a car in use fails. Compute the long-run average cost per unit time.
Note: In both (a) and (b) you are expected to compute the ratio of the expected cost incurred in a cycle to the expected time of a cycle. The answer should, of course, be the same in both parts.

Solution: **(a)** Clearly,

$$\mathbb{E}[\text{cost per cycle}] = C_1 - \overline{F}(A)R(A) + F(A)C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \int_0^A x\mathrm{d}F(x) + A(1-F(A))\,.$$

So, treating the cost as the reward, the renewal reward theorem gives

$$\lim_{t\to\infty}\frac{\mathbb{E}[\text{accumulated cost by t}]}{t} = \frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{time of cycle}]} = \frac{C_1 - \overline{F}(A)R(A) + F(A)C_2}{\int_0^A x\,dF(x) + A\overline{F}(A)}$$

(b) The chance that a car fails is $F(A)$, so the number, N, of cars bought between failures has the geometric distribution with parameter $p = F(A)$. We have,

$$\mathbb{E}[\text{cost per cycle}] = \mathbb{E}[NC_1 - (N-1)R(A) + C_2] = C_1/F(A) + (1 - 1/F(A))R(A) + C_2$$

and

$$\mathbb{E}[\text{time of cycle}] = \mathbb{E}[(N-1)A] + \mathbb{E}[\text{car life}|\text{car life} < A] = \overline{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A)\,.$$

Thus,

$$\lim_{t\to\infty} \frac{\mathbb{E}[\text{accumulated cost by t}]}{t} = \frac{C_1/F(A) + (1 - 1/F(A))R(A) + C_2}{\overline{F}(A)A/F(A) + \int_0^A x\,dF(x)/F(A)}\,.$$

Multiplying numerator and denominator by $F(A)$ gives the same expression as in (a).

Recommended reading:
Sections 3.4 through 3.7, excluding subsections 3.4.3, 3.6.1, 3.7.1. We will not cover the material in Section 3.8, though you may like to look through it.

Supplementary exercises: 3.24, 3.27, 3.35.
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.