Homework 5 (Stats 620, Winter 2017)

Due Tuesday Feb 21, in class
Questions are derived from problems in *Stochastic Processes* by S. Ross.

1. Prove that if the number of state is n, and if state j is accessible from state i, then it is accessible in n or fewer steps.

2. For states i, j, k with $k \neq j$, let

$$P_{ij/k}^n = P\{X_n = j, X_\ell \neq k, \ell = 1, ..., n-1 | X_0 = i\}.$$

   (a) Explain in words what $P_{ij/k}^n$ presents.
   (b) Prove that, for $i \neq j, P_{ij}^n = \sum_{k=0}^n P_{ii}^k P_{ij/i}^{n-k}$

3. Show that the symmetric random walk is recurrent in two dimensions and transient in three dimensions.

   Comments: This asks you to extend the argument of Ross Example 4.2(A) to two and three dimensions. You may use either of the definitions of the simple symmetric random walk in d dimensions from the notes.

4. A transition probability matrix P is said to be doubly stochastic if

$$\sum_i P_{ij} = 1 \quad \text{for all } j.$$

   That is, the column sums all equal 1. If a doubly stochastic chain has n states and is ergodic, calculate its limiting probabilities.

   Hint: guess the answer, and then show that your guess satisfies the required equations. Then, by arguing for the uniqueness of the limiting distribution, you will have solved the problem.

5. Jobs arrive at a processing center in accordance with a Poisson process with rate λ. However, the center has waiting space for only N jobs and so an arriving job finding N others waiting goes away. At most 1 job per day can be processed, and the processing of this job must start at the beginning of the day. Thus, if there are any jobs waiting for processing at the beginning of a day, then one of them is processed that day, and if no jobs are waiting at the beginning of a day then no jobs are processed that day. Let X_n denote the number of jobs at the center at the beginning of day n.
   (a) Find the transition probabilities of the Markov chain $\{X_n, n \geq 0\}$.
   (b) Is this chain ergodic? Explain.
   (c) Write the equations for the stationary probabilities.

   Instructions:
   (a). Suppose that the arrival rate λ has units day^{-1}.
   (b). You may assert the property that a finite state, irreducible, aperiodic Markov chain is ergodic (see Theorem 4.3.3, the discussion following this theorem, and Problem 4.14).
   (c). There is no particularly elegant way to write these equations, and you are not expected to solve them.

Recommended reading:
Sections 4.1 through 4.3, excluding examples 4.3(A,B,C).

Supplementary exercises: 4.13, 4.14
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.