Homework 6 (Stats 620, Winter 2017)

Due Thursday March 16, in class

1. In a branching process the number of offspring per individual has a Binomial $(2, p)$ distribution. Starting with a single individual, calculate:
   (a) the extinction probability;
   (b) the probability that the population becomes extinct for the first time in the third generation.
   (c) Suppose that, instead of starting with a single individual, the initial population size Z_0 is a random variable that is Poisson distributed with mean λ. Show that, in this case, the extinction probability is given, for $p > 1/2$, by

   $$\exp\{\lambda(1-2p)/p^2\}.$$

   Instructions: (b) If X_n is the size of the nth generation, this question is asking you to find $P(X_3 = 0, X_2 > 0, X_1 > 0 | X_0 = 1)$. This can be done by brute force calculation, or by using probability generating functions.

2. Consider a time-reversible Markov chain with transition probabilities P_{ij} and limiting probabilities π_i; and now consider the same chain truncated to the states $0, 1, , M$. That is, for the truncated chain its transition probabilities $\overline{P}_{ij}$ are

   $$\overline{P}_{ij} = \begin{cases} P_{ij} + \sum_{k>M} P_{ik}, & 0 \le i \le M, j = i \\ P_{ij}, & 0 \le i \ne j \le M \\ 0, & \text{otherwise.} \end{cases}$$

   Show that the truncated chain is also time reversible and has limiting probabilities given by

   $$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}$$

3. For an ergodic semi-Markov process:
   (a) Compute the rate at which the process makes a transition from i into j.
   (b) Show that $\sum_i P_{ij}/\mu_{ii} = 1/\mu_{jj}$.
   (c) Show that the proportion of time that the process is in state i and headed for state j is $P_{ij}\eta_{ij}/\mu_{ii}$ where $\eta_{ij} = \int_0^\infty \overline{F}_{ij}(t)\,dt$.
   (d) Show that the proportion of time that the state is i and will next be j within a time x is

   $$\frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x),$$

   where F_{ij}^e is the equilibrium distribution of F_{ij}

   Hint: all parts of this question can be done by defining appropriate renewal-reward processes. For (d), we use the definition $F_{ij}^e(x) = \int_0^x \bar{F}_{ij}(y)\,dy / \int_0^\infty \bar{F}_{ij}(y)$ (see Ross, p131). This is the delay required to make a delayed renewal process with renewal distribution F_{ij} stationary. It arises here since it is also the limiting distribution of the residual life process for a non-lattice renewal process.

4. A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean times between location i and j are $t_{12} = 20, t_{13} = 30$ and $t_{23} = 30$ (with $t_{ij} = t_{ji}$).
   (a) What is the (limiting) probability that the taxi's most recent stop was at location $i, i = 1, 2, 3$?
   (b) What is the (limiting) probability that the taxi is heading for location 2?
   (c) What fraction of time is the taxi traveling from 2 to 3? Note: Upon arrival at a location the taxi immediately departs.

Recommended reading:
Sections 4.5, 4.7, 4.8, 5.1, 5.2. You may skip Section 4.6, which will not be covered in this course.

Supplementary exercise: 4.40
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.