Homework 6 (Stats 620, Winter 2017)

Due Thursday March 16, in class

1. In a branching process the number of offspring per individual has a Binomial $(2, p)$ distribution. Starting with a single individual, calculate:
(a) the extinction probability;
(b) the probability that the population becomes extinct for the first time in the third generation.
(c) Suppose that, instead of starting with a single individual, the initial population size Z_0 is a random variable that is Poisson distributed with mean λ. Show that, in this case, the extinction probability is given, for $p > 1/2$, by

$$\exp\{\lambda(1-2p)/p^2\}.$$

Instructions: (b) If X_n is the size of the nth generation, this question is asking you to find $P(X_3 = 0, X_2 > 0, X_1 > 0 | X_0 = 1)$. This can be done by brute force calculation, or by using probability generating functions.

Solution:

(a) Say $p_j = \mathbb{P}[X_1 = j | X_0 = 1] = \binom{2}{j} p^j (1-p)^{2-j}$ for $j = 0, 1, 2$ and 0 for $j > 2$. Now $\pi_0 = \mathbb{P}[\text{Population dies out}] = \sum_{j=0}^{\infty} \pi_0^j p_j$. Thus

$$\pi_0 = (1-p)^2 + 2\pi_0(1-p)p + \pi_0^2 p^2 .$$

Solving and choosing the smaller root

$$\pi_0 = \begin{cases} 1 & p \leq .5 \\ (\frac{1-p}{p})^2 & p > .5 \end{cases}$$

(b) Let $\phi_n(s) = \mathbb{E}[s^{X_n}]$. It was shown in class that $\phi_n(s) = \phi_1(\phi_{n-1}(s))$. Also its easy to see that $\phi_1(s) = (sp + 1 - p)^2$. Also $\mathbb{P}[X_n = 0] = \phi_n(0)$. Thus the probability of extinction in third generation is

$$\begin{aligned} \phi_3(0) - \phi_2(0) &= \phi_1(\phi_1(\phi_1(0))) - \phi_1(\phi_1(0)) \\ &= 4p^2(1-p)^4 + 6p^3(1-p)^5 + 6p^4(1-p)^6 + 4p^5(1-p)^7 + p^6(1-p)^8 \end{aligned}$$

(c) Probability of extinction when $Z_0 = 1$ and $p > .5$ is $(\frac{1-p}{p})^2$. All families behave independently of each other. Thus when Z_0 has a Poisson distribution with parameter (λ), the probability of extinction equals

$$\sum_{n=0}^{\infty} \left[\frac{(1-p)^2}{p^2}\right]^n P[Z_0 = n] ,$$

which is exactly the probability generating function of Poisson r.v. evaluated at $\frac{(1-p)^2}{p^2}$. Thus probability of extinction equals $e^{\lambda[\frac{(1-p)^2}{p^2} - 1]}$.

2. Consider a time-reversible Markov chain with transition probabilities P_{ij} and limiting probabilities π_i; and now consider the same chain truncated to the states $0, 1, , M$. That is, for the truncated chain its transition probabilities $\overline{P}_{ij}$ are

$$\overline{P}_{ij} = \begin{cases} P_{ij} + \sum_{k>M} P_{ik}, & 0 \le i \le M, j = i \\ P_{ij}, & 0 \le i \ne j \le M \\ 0, & \text{otherwise.} \end{cases}$$

Show that the truncated chain is also time reversible and has limiting probabilities given by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}$$

<u>Solution</u>: Assume that the truncated chain is also irreducible. Simply verify that $\{\overline{\pi}_i\}_{0 \le i \le M}$ defined by

$$\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i},$$

satisfy

$$\overline{P}_{ij}\overline{\pi}_i = \overline{P}_{ji}\overline{\pi}_j \,, \forall 0 \le i, j \le M \quad \text{and} \quad \sum_{i=0}^{M} \overline{\pi}_i = 1\,..$$

Since the original Markov chain is time–reversible, we have

$$P_{ij}\pi_i = P_{ji}\pi_j \,, \forall i, j \ge 0\,.$$

It follows that for any $0 \le i, j \le M$, we have

$$\overline{P}_{ij}\overline{\pi}_i = \frac{\pi_i}{\sum_{i=0}^{M} \pi_i}\left(P_{ij} + \mathbf{1}_{\{i=j\}} \sum_{k>M}^{M} P_{ik}\right) = \frac{\pi_j}{\sum_{i=0}^{M} \pi_i}\left(P_{ji} + \mathbf{1}_{\{i=j\}} \sum_{k>M}^{M} P_{jk}\right) = \overline{P}_{ji}\overline{\pi}_j\,.$$

3. For an ergodic semi-Markov process:
(a) Compute the rate at which the process makes a transition from i into j.
(b) Show that $\sum_i P_{ij}/\mu_{ii} = 1/\mu_{jj}$.
(c) Show that the proportion of time that the process is in state i and headed for state j is $P_{ij}\eta_{ij}/\mu_{ii}$ where $\eta_{ij} = \int_0^\infty \overline{F}_{ij}(t)\, dt$.
(d) Show that the proportion of time that the state is i and will next be j within a time x is

$$\frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x),$$

where F_{ij}^e is the equilibrium distribution of F_{ij}

Hint: all parts of this question can be done by defining appropriate renewal-reward processes. For (d), we use the definition $F_{ij}^e(x) = \int_0^x \bar{F}_{ij}(y)\, dy / \int_0^\infty \bar{F}_{ij}(y)$ (see Ross, p131). This is the delay required to make a delayed renewal process with renewal distribution F_{ij} stationary. It arises here since it is also the limiting distribution of the residual life process for a non-lattice renewal process.

Solution: **(a)** Define a (delayed) renewal reward process: a renewal occurs when state i is entered from other states and the reward of each n–th cycle R_n equals 1 if in the n–th cycle, the state after i is j and 0 otherwise. Let $R_{ij}(t)$ be the total number of transitions from i to j by time t. We have

$$\sum_{n=0}^{N(t)} R_n \leq R_{ij}(t) \leq \sum_{n=0}^{N(t)+1} R_n \leq \sum_{n=0}^{N(t)} R_n + 1\,.$$

Thus the rate at which the process makes a transition from i to j equals

$$\lim_{t\to\infty} \frac{R_{ij}(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}}{\mu_{ii}}\,.$$

(b) Let $R_j(t)$ be the number of visits to state j by time t. Thus

$$\begin{aligned} \sum_i R_{ij}(t) &= R_j(t) \\ \sum_i \lim_{t\to\infty} \frac{R_{ij}(t)}{t} &= \lim_{t\to\infty} \frac{R_j(t)}{t} \\ \sum_i \frac{P_{ij}}{t} &= \frac{1}{\mu_{jj}} \end{aligned}$$

(c) Define cycle as in part (a) and the reward in a cycle to be 0 if the transition from i is not into j and T_{ij} the time taken for transition if the transition from i is into j. Thus

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}\mathbb{E}[T_{ij}]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}}\,.$$

(d) Define cycle as in last part and the reward in a cycle as 0 if the transition from i is not into j and $\min(x, T_{ij})$ if the transition from i is into j. Thus

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{P_{ij}\mathbb{E}[\min(x, T_{ij})]}{\mu_{ii}} = \frac{P_{ij}\eta_{ij}}{\mu_{ii}} F_{ij}^e(x)\,.$$

4. A taxi alternates between three locations. When it reaches location 1 it is equally likely to go next to either 2 or 3. When it reaches 2 it will next go to 1 with probability 1/3 and to 3 with probability 2/3. From 3 it always goes to 1. The mean times between location i and j are $t_{12} = 20, t_{13} = 30$ and $t_{23} = 30$ (with $t_{ij} = t_{ji}$).
(a) What is the (limiting) probability that the taxi's most recent stop was at location $i, i = 1, 2, 3$?
(b) What is the (limiting) probability that the taxi is heading for location 2?
(c) What fraction of time is the taxi traveling from 2 to 3? Note: Upon arrival at a location the taxi immediately departs.

Solution: First we write the transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 0 & 0 \end{bmatrix}\,.$$

Now the stationary probabilities can be found by solving $\pi = \pi P$. We have

$$\pi_1 = \frac{6}{14}\,, \pi_2 = \frac{3}{14}\,,\ \text{and}\ \pi_3 = \frac{5}{14}\,.$$

Since $\mu_i = \sum_i P_{ij}\mu_{ij}$, we have $\mu_1 = 25$, $\mu_2 = 80/3$ and $\mu_3 = 30$.

(a) By formula

$$P_i = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j}\,,$$

we have

$$P_1 = \frac{15}{38}\,, P_2 = \frac{8}{38}\,,\ \text{and}\ P_3 = \frac{15}{38}\,.$$

They are the correspondingly required limiting probabilities.

(b) Use part (c) of 4.48. Since the taxi can only go to location 2 from location 1, the limiting probability that taxi is headed for location 2 equals

$$P_{12}\eta_{12}(\frac{\mu_1}{P_1})^{-1} = \frac{3}{19}\,.$$

(c) Same argument as in part (b) implies that the proportion of the time that the taxi is traveling from location 2 to location 3 equals

$$P_{23}\eta_{23}(\frac{\mu_2}{P_2})^{-1} = \frac{3}{19}\,.$$

Recommended reading:
Sections 4.5, 4.7, 4.8, 5.1, 5.2. You may skip Section 4.6, which will not be covered in this course.

Supplementary exercise: 4.40
These are optional, but recommended. Do not turn in solutions—they are in the back of the book.