Homework 7 (Stats 620, Winter 2017)

Due Thursday March 23, in class

1. Show that a continuous-time Markov chain is regular, given (a) that $\nu_i < M < \infty$ for all i or (b) that the corresponding embedded discrete-time Markov chain with transition probabilities P_{ij} is irreducible and recurrent.

   Hint: For (a), you may follow the method suggested in the book solution (p. 491).

2. Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain on the non-negative integers, having transition rates q_{ij}. Let $P(t) = P_{00}(t)$.

   (a) Find $\lim_{t\to 0} \frac{1-P(t)}{t}$.

   (b) Show that $P(t)P(s) \leq P(t+s) \leq 1 - P(s) + P(s)P(t)$.

   (c) Show $|P(t) - P(s)| \leq 1 - P(t-s)$, $s < t$ and conclude that P is continuous.

   Hint: For (a) you should justify your answer but need not prove the necessary limit theorem, so your answer could be quite short! One way to obtain (b) is through two applications of the Chapman-Kolmogorov identity. One way to solve (c) is by algebraic manipulation of (b).

3. Suppose that the "state" of a system can be modeled as a two-state continuous-time Markov chain with transition rates $\nu_0 = \lambda, \nu_1 = \mu$. When the state of the system is i, "events" occur in accordance with a Poisson process with rate α_i for $i = 0, 1$. Let $N(t)$ denote the number of events in $(0, t)$.

   (a) Find $\lim_{t\to\infty} N(t)/t$.

   (b) If the initial state is state 0, find $\mathbb{E}[N(t)]$.

   Hint For (a), one approach is to let return times into state 0 form a renewal process, and consider a reward to be the number of "events" in the renewal period. For (b), you are asked to find the exact result for finite t, rather than a limiting result as $t \to \infty$.

4. Consider a population in which each individual independently gives birth at an exponential rate λ and dies at an exponential rate μ. In addition, new members enter the population in accordance with a Poisson process with rate θ. Let $X(t)$ denote the population size at time t.

   (a) Explain why $\{X(t), t \geq 0\}$ is a birth/death process. What are its parameters?

   (b) Set up and solve a differential equation to find $\mathbb{E}\big[X(T)|X(0) = i\big]$.

Recommended reading:
Sections 5.3, 5.4, 5.5.

Supplementary exercise: 5.14
Optional, but recommended. Do not turn in a solution—it is in the back of the book.