Homework 8 (Stats 620, Winter 2017)

Due Thursday March 30, in class

1. Let $\{N(t)\}$ be a Poisson process with rate λ. Since $\{N(t)\}$ is also a continuous time Markov chain, it can also be defined in terms of its transition rates q_{ij} and rates ν_i of leaving i (together with its initial distribution $P_i(0) = \mathbb{P}\{N(0) = i\}$). For some fixed time T, let $N^*(t) = N(T-t)$ for $0 \le t \le T$. $\{N^*(t)\}$ is an inhomogeneous continuous time Markov chain, so it is specified by time-dependent parameters $q^*_{ij}(t)$ and $\nu^*_i(t)$ (together with its initial distribution $P^*_i(0) = \mathbb{P}\{N^*(0) = i\}$).

   (a) Write down expressions for q_{ij}, ν_i and $P_i(0)$.

   (b) Obtain expressions for $q^*_{ij}(t)$, $\nu^*_i(t)$ and $P^*_i(0)$. Note that q^*_{ij} does not depend on λ. Note also that duration in state i viewed in reverse time is different from forward in time (as mentioned in the notes, they are the same if the Markov chain is stationary).

   Hint: One approach is to employ Theorem 2.3.1 of Ross, which you may use without proof.

2. The following problems arises in molecular biology. The surface of a bacterium consists of several sites at which foreign molecules—some acceptable and some not—become attached. We consider a particular site and assume that molecules arrive at the site according to a Poisson process with parameter λ. Among these molecules a proportion α are acceptable. Unacceptable molecules stay at the site for a length of time which is exponentially distributed with parameter μ_1, whereas an acceptable molecule remains at the site for an exponential time with departure rate μ_2. An arriving molecule will become attached only if the site is free of other molecules.

   (i) What percentage of the time is the site occupied with an acceptable molecule?

   (ii) What fraction of arriving acceptable molecules become attached?

3. An undirected graph has n vertices and edges between all $n(n-1)/2$ vertex pairs. A particle moves along the graph as follows: Events occur along the edge (i, j) according to independent Poisson processes with rates λ_{ij}. An event on edge (i, j) causes the edge to become "excited". If the particle is at vertex i the moment that the edge (i, j) becomes excited then the particle instantaneously moves from to vertex j. Let P_j denote the (limiting) proportion of the time that the particle is at vertex j. Explain why the position of the particle follows a continuous time Markov chain, and hence show that $P_j = 1/n$.

   Hint: use time reversibility.

4. Verify that $X_n/m^n, n \ge 1$, is a martingale when X_n is the size of the n^{th} generation of a branching process whose mean umber of offspring per individual is m.

5. Consider the Markov chain which at each transition either goes up 1 with probability p or down 1 with probability $q = 1 - p$. Argue that $(q/p)^{S_n}$, $n \ge 1$, is a martingale.

6. Consider a Markov chain $\{X_n, n \geq 0\}$ with $P_{NN} = 1$. Let $P(i)$ denote the probability that this chain eventually enters state N given that it starts in state i. Show that $\{P(X_n), n \geq 0\}$ is a martingale.

   Hint: One approach involves defining $A = \{X_\infty = N\}$ and showing that $\mathbb{E}\big[\mathbb{P}(A \mid X_n) \mid X_{n-1}] = \mathbb{E}\big[\mathbb{P}(A \mid X_n, X_{n-1}) \mid X_{n-1}\big] = \mathbb{P}(A \mid X_{n-1})$.

Recommended reading:
Sections 5.6 (up to Prop. 5.6.3, i.e. pp 257–261), 6.1, 6.2.

Supplementary exercise: 6.2, 6.7
Optional, but recommended. Do not turn in solutions.